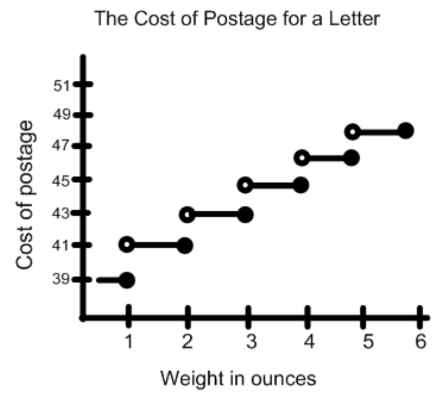
Chapter 1



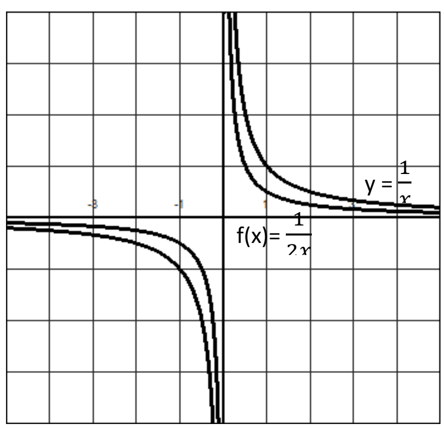


Figure 1: The US Unemployment Rate since 1990

Income $	Federal Tax Rate
0-8699	10%
8700-35349	15%
35350-85649	25%
85650-178649	28%
178650-388349	33%
388350-500000	35%

1.1 Functions and Function Operations

A function is a rule which assigns exactly one unique output value to each input value. Functions may be defined by a table, a graph, a formula, or verbally. **Note:** Different input values may have the same output value.

Numerically: Check that each unique input value in the table has exactly one output value associated with it.

Examples:

Determine whether or not each of the following tables represents a function.

1.

Age (years)	Height (inches)
5	40
9	45
13	60
17	66

For each age, there is one height associated with that age. This table represents a function.

2.

Time (seconds)	Height (feet)
1	3
2	51
3	67
4	51

This table represents a function. For each time there is exactly one height. An object can't be at two different heights at the same time.

3.

Federal Tax Rate	Income $
10%	0-8699
15%	8700-35349
25%	35350-85649
28%	85650-178649
33%	178650-388349
35%	388350+

This table does not represent a function because a tax rate of 10% has many possible income amounts. If one input has more than one output then it is not a function.

4.

x	y
0	6
-6	0
0	-6
6	0

This table does not represent a function because an input value of x = 0 results in two output values of 6 and -6.

Graphically:

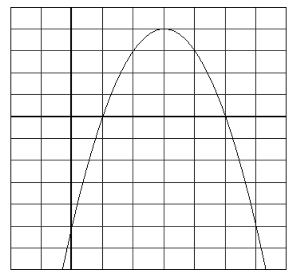
Vertical Line test: If a vertical line can be drawn that crosses the graph in more than one point, then the graph does not represent a function.

Note: All you have to find is one vertical line that touches the graph in more than one point and the graph does not represent a function.

Examples:

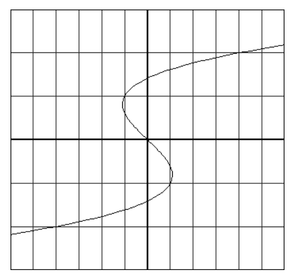
1. Is the graph shown the graph of a function?

   Yes, every vertical line that can be drawn will only cross the graph at one point. This graph passes the vertical line test.



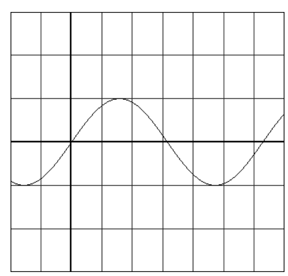
2. Is the graph shown the graph of a function?

   No, a vertical line drawn at x = 0 will cross the graph in three points (more than one point). This graph fails the vertical line test.



3. Is the graph shown the graph of a function?

   Yes, every vertical line that can be drawn will only cross the graph at one point. This graph passes the vertical line test.



Algebraically: For every input value, the equation should only produce one output value. To test if the equation represents a function, replace the input variable with a value, solve for the output value or values.

Examples:

1. Is $y = 2x^2 + x - 3$ a function?

   Yes, if you replace x with any value then there will be exactly one value of y. For example, when $x = 2$, $y = 2(2)^2 + 2 - 3 = 7$.

2. Is $x^2 + y^2 = 9$ a function?

   Replacing x = 0 yields $y^2 = 9$ and solving for y gives that y can be equal to both -3 and 3. Therefore, one input value corresponds to two output values and this equation does not represent a function.

Verbally: First determine which quantity represents the input and the output. Then, determine whether it is possible to have more than one output value for a single input value.

Examples:

1. Is a student's current class schedule a function of the student?

   Yes, a single student will only have one schedule of classes for this term.

2. If the variables were interchanged in the example above, would it still represent a function?

   No, more than one student might have the same schedule during a term. So, one schedule could correspond to more than one student.

3. Is the height of an object a function of time?

   Yes, an object cannot be at more than one height at the same time.

Function notation: If a relation between variables is a function, then the relation can be expressed using function notation. Function notation has a form of f(input)=output. The letter f (which can be any letter) is the name of the function. The notation gives a compact way to write the relationship between the input and output. To use function notation, replace the input variable with the given value and evaluate to find the output.

Examples:

1. Given the function $f(x) = 3x + 5$, find the following.

   A. f(2)

      To find f(2), the value of x is replaced with 2 yielding $f(2) = 3(2) + 5 = 11$. Therefore, $f(2) = 11$.

   B. f(5d)

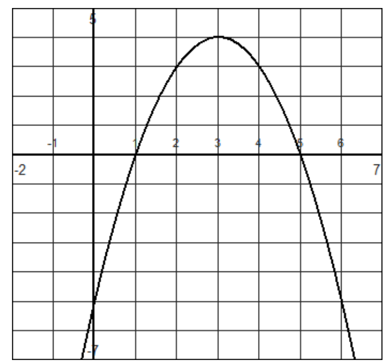
      $f(5d) = 3(5d) + 5$ so $f(5d) = 15d + 5$

2. The table shown gives the height of an individual at certain ages. Use the table to find H(13).

Looking at the table and finding where A = 13 gives H(13) = 60. This means that the individual was 60 inches tall at age 13.

A, years	H(A), inches
5	40
9	45
13	60
17	66

3. Given the graph of g(x) as shown to the right and assuming the scales on both axes are 1 unit, evaluate the following.



A. g(1)

Finding the point of the graph which has an x-value of 1, we find the y-value of that point. The graph contains the point (1, 0) so when x = 1 then g(1) = 0.

B. g(3)

Looking at where x = 3, the point on the graph is at (3, 4) so g(3) = 4.

Combining Functions

Functions can be combined to make new functions using mathematical operations. They may be added, subtracted, multiplied, divided and composed.

Operations on functions

If f(x) and g(x) both exist, then the sum, difference, product and quotient are defined as shown:

(f + g)(x) = f(x) + g(x)

(f – g)(x) = f(x) – g(x)

(fg)(x) = f(x) · g(x)

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where g(x) ≠ 0

Examples:

1. Given f(x) = 3x – 7 and g(x) = 5x + 2, find (f + g)(x), (f – g)(x), (fg)(x) and $\left(\frac{f}{g}\right)(x)$.

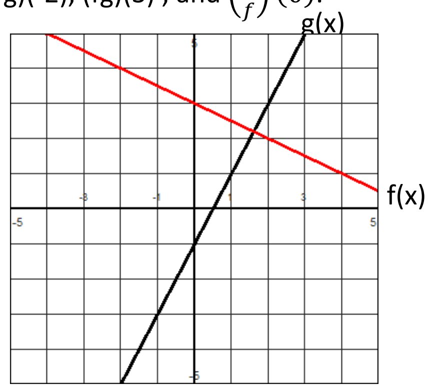
(f + g)(x) = f(x) + g(x) = (3x – 7) + (5x + 2) = 8x – 5

$(f - g)(x) = f(x) - g(x) = (3x - 7) - (5x + 2) = -2x - 9$

$(fg)(x) = f(x) \cdot g(x) = (3x - 7)(5x + 2) = 15x^2 - 29x - 14$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{3x-7}{5x+2}$

2. Given the graphs of f(x) and g(x) as shown, find (f + g)(-2), (fg)(3) , and $\left(\frac{g}{f}\right)(0)$.



$(f + g)(-2) = f(-2) + g(-2) = 4 + -5 = -1$

$(fg)(3) = f(3) \cdot g(3) = (1.5)(5) = 7.5$

$\left(\frac{g}{f}\right)(0) = \frac{g(0)}{f(0)} = \frac{-1}{3}$

3. Given $f(x) = x^2 - 3$, find f(a + h), f(a + h) – f(a), and $\frac{f(a+h)-f(a)}{h}$.

$f(a + h) = (a + h)^2 - 3 = a^2 + 2ah + h^2 - 3$

$f(a + h) - f(a) = [(a + h)^2 - 3]-[a^2 - 3] = [a^2 + 2ah + h^2 - 3] - [a^2 - 3] = 2ah + h^2$

$\frac{f(a+h)-f(a)}{h} = \frac{2ah+h^2}{h} = \frac{h(2a+h)}{h} = 2a + h$

Composition of functions

If f and g are functions, then the composite function f ∘ g, or the composition of f and g is defined by

(f ∘ g)(x) = f(g(x)) and is read f of g of x. The domain of the composition of two functions consists of numbers which are in the domain of the inside function, g(x) and also in the domain of f(g(x)).

Examples:

1. Given $f(x) = 2x - 5$ and $g(x) = x^2 - 3$, find (f ∘ g)(x) and (g ∘ f)(x). Find the domain of each composite function.

$(f \circ g)(x) = f(g(x)) = f(x^2 - 3) = 2(x^2 - 3) - 5 = 2x^2 - 11$

Since both $g(x) = x^2 - 3$ and $(f \circ g)(x) = 2x^2 - 11$ have a domain of all real numbers, then the domain of the composite function is all real numbers.

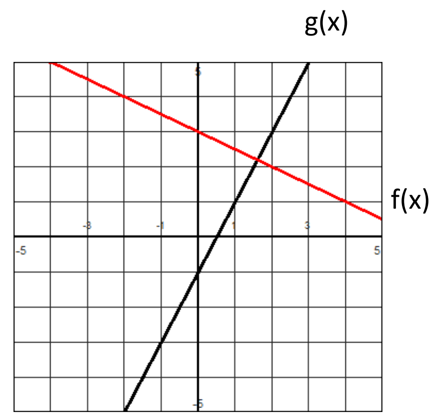
$(g \circ f)(x) = g(f(x)) = g(2x - 5) = (2x - 5)^2 - 3 = 4x^2 - 20x + 25 - 3 = 4x^2 - 20x + 22$

Since both $f(x) = 2x - 5$ and $(f \circ g)(x) = 4x^2 - 20x + 22$ have a domain of all real numbers, then the domain of the composite function is all real numbers.

2. Use the graphs shown to find $(f \circ g)(-1)$ and $(g \circ f)(4)$.

   $(f \circ g)(-1) = f(g(-1)) = f(-3) = 4.5$

   $(g \circ f)(4) = g(f(4)) = g(1) = 1$



3. Use the table shown to find, $(f - g)(3)$, $(fg)(1)$ and $(f \circ g)(0)$.

   $(f - g)(3) = f(3) - g(3) = 3 - 1 = 2$

   $(fg)(1) = f(1) \cdot g(1) = (5)(4) = 20$

   $(f \circ g)(0) = f(g(0)) = f(2) = -1$

x	f(x)	g(x)
0	7	2
1	5	4
2	-1	0
3	3	1
4	0	9

4. Find the domain of $(f \circ g)(x)$ when $f(x) = \frac{1}{x-1}$ and $g(x) = \sqrt{x}$.

   The domain of g(x) is $[0, \infty)$.

   $(f \circ g)(x) = \frac{1}{\sqrt{x}-1}$ The domain of this function will consist of numbers except for those which make the denominator equal to zero. The denominator will be equal to zero at x = 1. Therefore, the domain of the composite function is $[0, 1) \cup (1, \infty)$.

Decomposition of Functions

To decompose a function means to find the two functions that when composed will make up the given function. We need to find an f(x) and g(x) such that $(f \circ g)(x) = h(x)$, where h(x) is the given function. The concept is to look for an inside function, g(x), and an outside function, f(x). This is usually easy to do if there are parentheses in the function. The inside function will be inside the parentheses. In other functions it may be more difficult to see the decomposition. Keep in mind that there may be more than one way to decompose a function. The overall purpose is to write a complicated function in terms of its simpler parts.

Examples:

1. Find f(x) and g(x) such that $(f \circ g)(x) = h(x)$ where $h(x) = (3x + 1)^4$.

   Given $h(x) = (3x + 1)^4$, we can identify an inside function as the terms inside the parentheses which would give $g(x) = 3x + 1$. Then, since these terms are raised to a power, the outside function would be $f(x) = x^4$. When composed, these two functions will give $(3x + 1)^4$.

2. Find f(x) and g(x) such that $(f \circ g)(x) = h(x)$ where $h(x) = |x^2 - x + 2|$.

   Given $h(x) = |x^2 - x + 2|$, we can see that $x^2 - x + 2$ is inside the absolute value signs so this is what we would choose for g(x). The outside function would be the absolute value function. So, $f(x) = |x|$ and $g(x) = x^2 - x + 2$.

3. Find f(x) and g(x) such that $(f \circ g)(x) = h(x)$ where $h(x) = 4^{x^3}$.

   In this case, the exponent of the function would be chosen for the inside function and the exponential equation would be the outside function. Therefore, $f(x) = 4^x$ and $g(x) = x^3$.

1.1 Homework:

1. Are the following functions? Explain why.

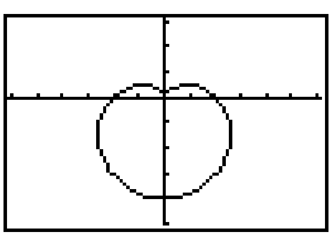
A.

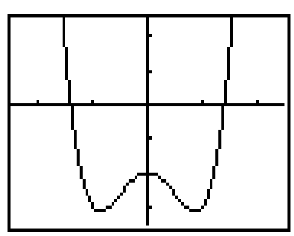
x	w
1	7
3	7
4	-1
8	3
-3	-1

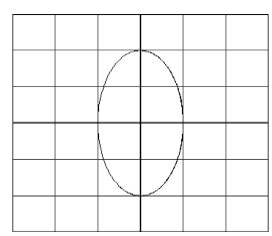
B.

t	h
-3	5
-2	6
-1	7
-2	8
-3	9

2. Are the following functions? Explain why.

A. 

B. 

C. 

3. Is temperature in Celsius a function of temperature in Fahrenheit?

4. Is postage a function of weight of a letter?

5. Is starting salary a function of years of schooling?

6. Is $3x + 5y = 12$ a function?

7. Is $x^2 + 2y^2 = 24$ a function?

8. Is $x + |y| = 6$ a function?

Evaluate the following functions.

9. Given $F(x) = 5x^3 - 8x + 1$, find

A. F(-2)

B. F(4)

C. $F(h^2)$

D. F(t) – 3

10. Given $F(t) = \frac{t+1}{2t-3}$, find:

A. F(2)

B. F(n + 1)

C. F(2g) + 1

D. 3F(a)

11. Given $g(t) = 3t^2 - 5t$, find:

A. g(-4)

B. g(3)

C. g(a) – 11

D. g(x + h)

12. Use the table shown to the right to find:

A. g(4)

B. g(10)

C. For what t-value is g(t) = 11?

t	g(t)
2	11
3	-7
4	-1
8	5
10	9

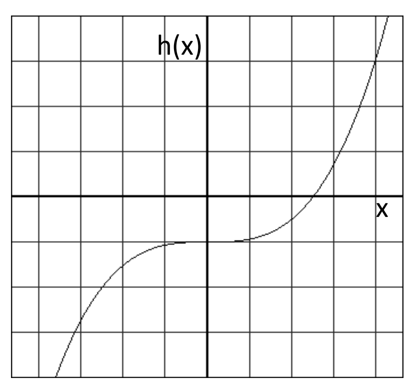
13. Assume the scales are 1 unit on both axes. Approximate the following function values from the graph.

A. h(0)

B. h(4)

C. h(-2)

D. For what x-value(s) is h(x) = -2?



14. Given f(x) = 6x – 1 and g(x) = x + 7, find:

A. (f + g)(x) B. (f – g)(x)

C. (fg)(4) D. $\left(\frac{f}{g}\right)(x)$

E. (f ∘ g)(x) F. (g ∘ f)(-2)

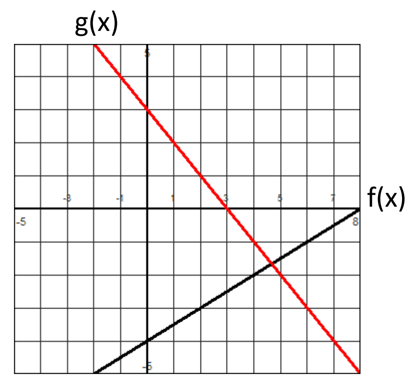
15. Given f(x) = x^2 and g(x) = 2x + 7, find:

A. (f + g)(3) B. (f – g)(x)

C. (fg)(x) D. $\left(\frac{f}{g}\right)(x)$

E. (f ∘ g)(1) F. (g ∘ f)(x)

16. Given the graph shown, find the following:



A. (f + g)(4) B. (f – g)(1)

C. (fg)(-1) D. $\left(\frac{f}{g}\right)(1)$

E. (f ∘ g)(3)

17. Given the table shown, find the following:

x	-1	0	1	2	3	4	5
f(x)	5	4	3	2	1	0	-1
g(x)	9	5	1	-3	-7	-11	-15

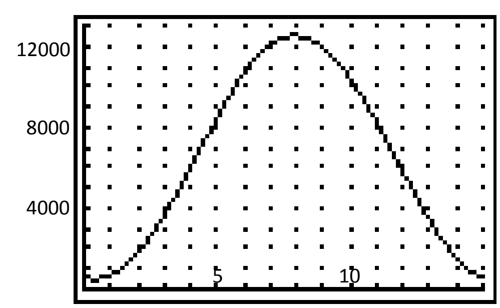
A. (f + g)(3) B. (g – f)(1)

C. (fg)(-1) D. (g ∘ f)(3)

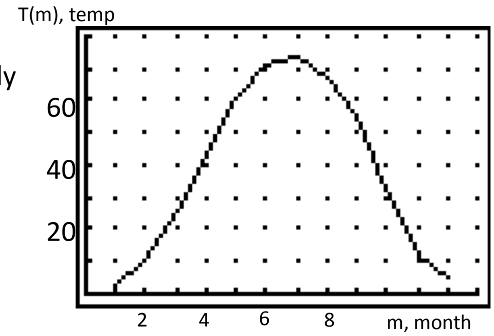
18. The graph shows S as a function of w. S represents the weekly sales of a best-selling book w weeks after it is released.

A. Find S(3) and explain what this represents in the context of the problem.
B. In which weeks were sales over $8000?

C. Approximately, what week had the highest sales? What were the sales that week?



19. The graph shows the graph of the average high temperatures in Fairbanks, Alaska for January (month 1) through December (month 12).



A. What month has the highest average temperature? Approximately what is the temperature during that month? Write this point in function notation.

B. Estimate T(4) and explain what this means in context.

C. During which months is the average high temperature below 30 degrees?

For 20-25, find f(x) and g(x) so that h(x) = $(f \circ g)(x)$.

20. h(x) = $\sqrt{3x^2 + 4}$

21. h(x) = $(9x - 11)^{5/3}$

22. h(x) = $|2x - 1|$

23. h(x) = 2^{4x+1}

24. h(x) = $\dfrac{1}{(2x+3)^2}$

25. h(x) = $\dfrac{x^2 + 1}{x^2}$

26. For f(x) = 4x + 7, find:

A. f(a +h) B. f(a+h) – f(a) C. $\frac{f(a+h)-f(a)}{h}$

27. For f(x) = x^2 + x, find:

A. f(a +h) B. f(a+h) – f(a) C. $\frac{f(a+h)-f(a)}{h}$

28. For f(x) = $\frac{1}{x}$, find:

A. f(a +h) B. f(a+h) – f(a) C. $\frac{f(a+h)-f(a)}{h}$

29. For f(x) = x^3 – 1, find:

A. f(a +h) B. f(a+h) – f(a) C. $\frac{f(a+h)-f(a)}{h}$

30. Find the domain of (f ∘ g)(x) when f(x) = $\frac{1}{x-2}$ and g(x) = $\frac{1}{x}$.

31. Find the domain of (f ∘ g)(x) when f(x) = x^3 + 3 and g(x) = $\sqrt{x}$.

1.2 Domain and Range

The **domain** of a function consists of all real numbers for which a function is defined. In other words, the domain is all possible values for the input of the function.

The **range** of a function consists of all the possible output values for the function corresponding to the domain values.

Numerically: When a function is given numerically, the domain is all the input values in the table and the range is all the output values in the table.

Examples:

1. Find the domain and range of the table shown listing the number of credit hours taken by certain students.

Name	Number of Credit Hours
Matthew	15
Melvin	12
Mary	15
Michael	10

The domain of this function is the set of names {Matthew, Melvin, Mary, Michael}. The range is the set of output values which are {10, 12, 15}. Note that if an output value is listed more than once in the table it only needs to be listed once in the range.

2. The table lists the 2013 federal tax rate for incomes between $0 and $500,000. Find the domain and range.

Income $	Federal Tax Rate
0-8699	10%
8700-35349	15%
35350-85649	25%
85650-178649	28%
178650-388349	33%
388350-500000	35%

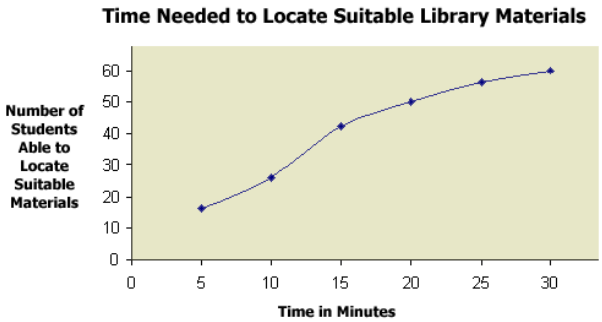
The domain of this of this function is the interval of incomes so the domain is [0, 500000] and the range of this function consists of the corresponding output values for the federal tax rate which are {10, 15, 25, 28, 33, 35}.

Graphically: To identify the domain of a function graphically, find the values for the input variable which have points on the graph. Looking in a left to right direction, identify the values where the graph is defined.

To identify the range of a function graphically, find the values of the output variable where the graph is defined. Look from the bottom to the top of the graph.

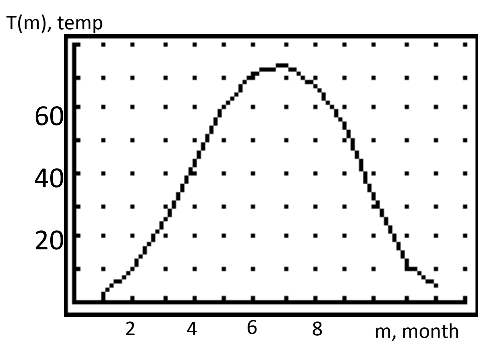
Examples:

1. The graph shows the number of students who can find suitable research materials given a particular amount of time in the library. Estimate the domain and range.



The domain of the graph consists of all values for time in minutes where the graph exists. The domain is [5, 30]. The range is all values for the number of students able to locate suitable materials and is approximately [18, 60].

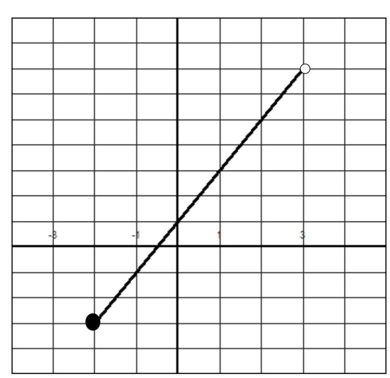
2. The graph shown represents the average high temperature in degrees Fahrenheit in Alaska from January to December where January is month 1. Estimate the domain and range.



The domain of the graph are the months shown which are [1, 12]. The range is the resulting average high temperatures which are approximately from [0, 73].

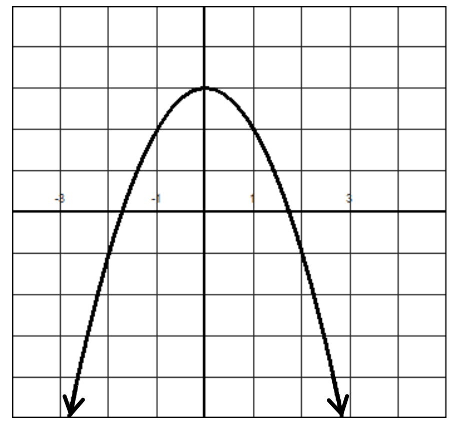
3. Given the graph shown, find the domain and range.

The domain is [-2, 3). The range is [-3,7). Note the use of parentheses and brackets to indicate whether or not an endpoint is included in the interval.



4. Given the graph, find the domain and range.

   The domain of the graph is all real numbers since the graph continues infinitely from left to right. Thus, the domain is (-∞, ∞). If we look at the graph from bottom to top, the range is (-∞, 3].



Algebraically: To find the domain of a formula, we assume the domain is as large as possible. We include all input values that make sense when substituted into the formula. We will not include any values which make the formula undefined such as numbers which cause a division by zero or numbers which result in the square root of negative numbers. The range consists of the output values for the domain.

Examples:

1. Given f(x) = $\frac{1}{x-5}$, find the domain and range.

   Because division by zero is not permissible, we must omit all values for which the denominator is equal to zero. Since x – 5 = 0 when x = 5, we exclude x = 5 from the domain. The function is defined for all other real numbers so the domain is all real numbers except x = 5. This can also be written in interval notation as (-∞, 5) U (5, ∞). The range consists of all values that can be outputs. As the fraction $\frac{1}{x-5}$ can never be equal to zero, the range is all numbers except for zero. This is written as (-∞, 0) U (0, ∞).

2. Given g(x) = $\sqrt{4-x^2}$, find the domain and range.

   In order to be defined, the radicand must be greater than or equal to zero since the square root of a negative number is not a real number. Therefore, $4 - x^2$ must be greater than or equal to zero. This occurs for values of x in the interval [-2, 2] so that is the domain of this function. The square root of a number must be greater than or equal to zero; thus, the range is [0, ∞).

Restricting the domain: In application problems, we often restrict the domain of the function to fit the situation. This occurs for many variables that may only have positive values such as time. When the domain has restrictions, the range is often restricted too.

Example: The function $h(t) = 784 - 16t^2$ gives the height of an object dropped from the top of a 784 foot building. Find a suitable domain and the corresponding range.

The height of the object must be greater than or equal to zero since ground level is at a height of zero feet. Setting the equation equal to zero, we get $784 – 16t^2 = 0$. Solving for t, we find that t = 7 seconds. Time can't be negative so the domain is [0, 7] seconds. The object is simply dropped from the top of the building so the maximum height is the starting height of 784 feet and the minimum height is 0 feet (when it hits the ground) so the range is [0, 784].

1.2 Homework:

Find the domain and range of each function. Approximate if necessary.

1.

x	F(x)
0	7
1	9
2	11
3	13
4	15

2.

Destination	Shipping Time
Southeast US	2 business days
Northeast US	3 business days
Midwest US	3 business days
Southwest US	3 business days
Western US	4 business days

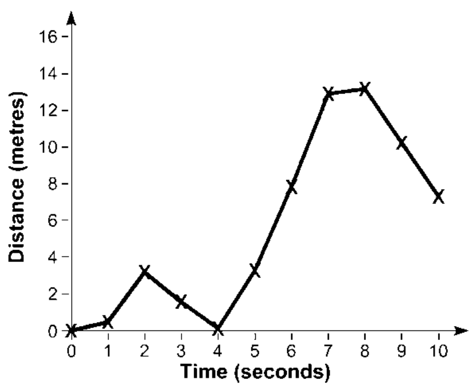
3.

Order Total	Standard
$0-$200	$23.00
$201-$300	$39.00
$300+	$99.00

4.

t	g(t)
-1	11
3	-7
4	-1
8	5
10	-1

5.



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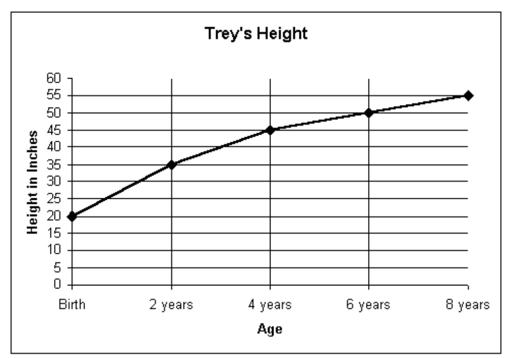

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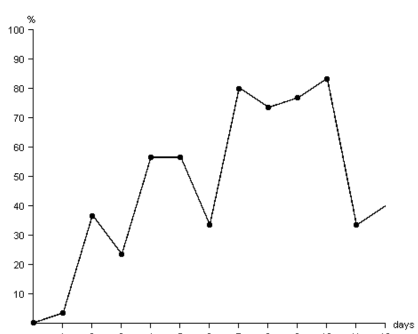
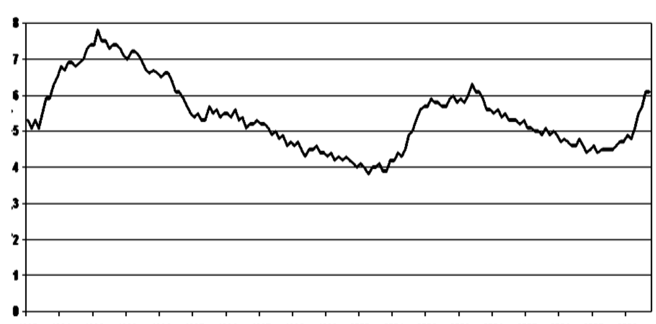


8.

Figure 1: The US Unemployment Rate since 1990



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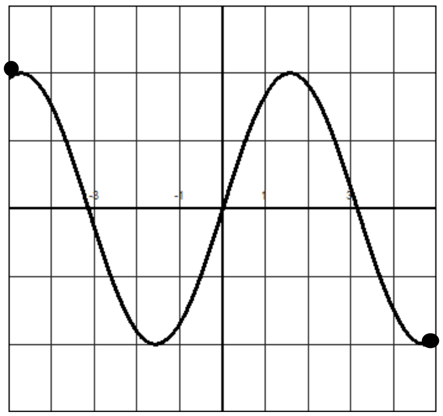

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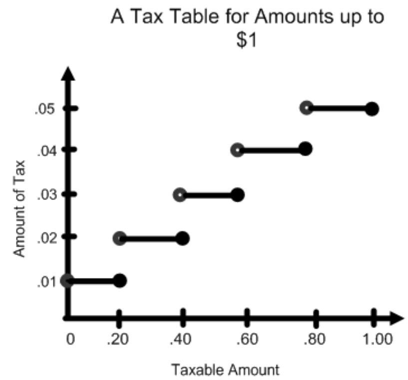

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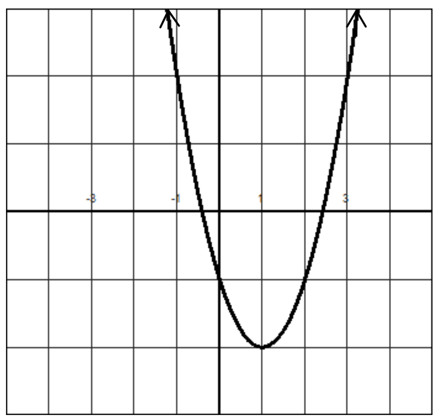

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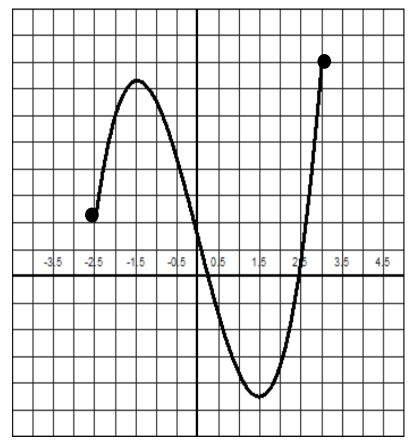
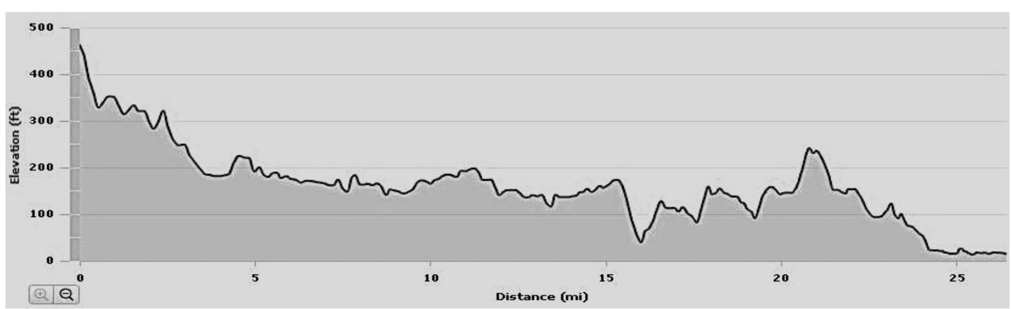


13. The graph shows the elevation in relation to the distance run in the Boston marathon.



14. $f(x) = 2x + 7$

15. $g(x) = 3x^2 - 1$

16. $h(x) = \frac{2}{x+3}$

17. $f(x) = \frac{1}{(x+2)(x-1)}$

18. $g(x) = \sqrt{2x+5}$

19. $h(x) = \sqrt{25-x^2}$

20. Use your calculator to graph $f(x) = 4x^3 - x^2$ on the domain [-3, 2], what is the corresponding range?

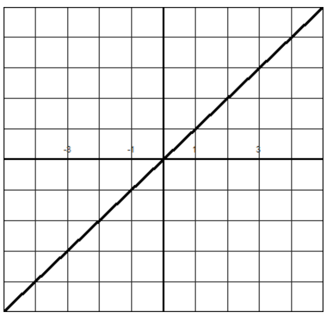
21. Use your calculator to graph $f(x) = 2x^2 + x - 1$ on the domain [-1, 3], what is the corresponding range?

1.3 Basic Shapes of Graphs

In algebra, there are 10 basic functions that are used for modeling. Eight of these functions are covered in this section and the others will be covered later in the course. Circles will also be covered in this section but circles are not functions.

Graphs of the eight basic shapes for functions are shown including their domain and range. You should learn the basic shape and a few points on each graph. The points you should be sure to plot when graphing the basic shapes are listed beside each graph. Plotting certain points helps to make sure the graphs are the correct width.

Linear function: f(x) = x

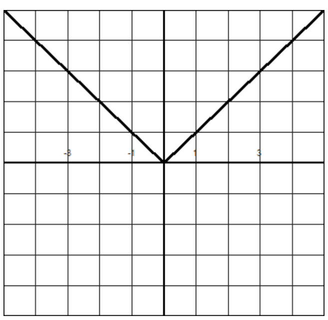


Domain: (-∞, ∞)

Range: (-∞, ∞)

Points: (-1, -1), (0, 0), (1, 1)

Absolute value function: f(x) = $|x|$

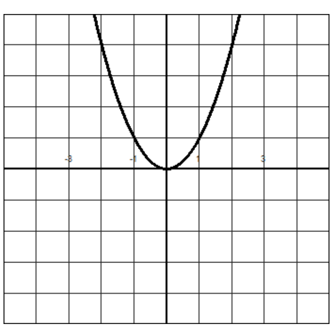


Domain: (-∞, ∞)

Range: [0, ∞)

Points: (-1, 1), (0, 0), (1, 1)

Quadratic function: f(x) = x^2

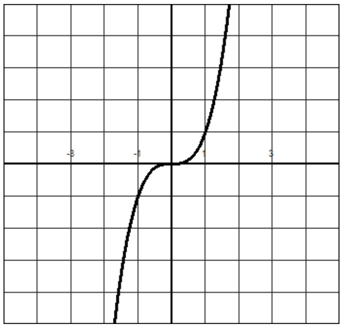


Domain: (-∞, ∞)

Range: [0, ∞)

Points: (-1, 1), (0, 0), (1, 1)

Cubic function: f(x) = x^3

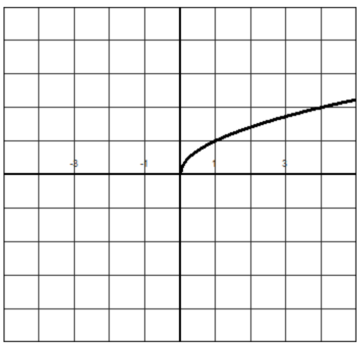


Domain: (-∞, ∞)

Range: (-∞, ∞)

Points: (-1, -1), (0, 0), (1, 1)

Square root function: f(x) = $\sqrt{x}$

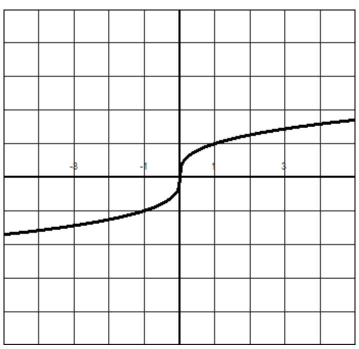


Domain: [0, ∞)

Range: [0, ∞)

Points: (0, 0), (1, 1)

Cube root function: f(x) = $\sqrt[3]{x}$

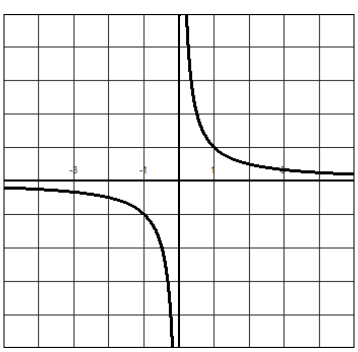


Domain: (-∞, ∞)

Range: (-∞, ∞)

Points: (-1, -1), (0, 0), (1, 1)

Rational function: f(x) = $\frac{1}{x}$

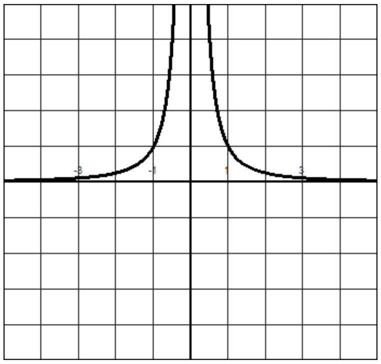


Domain: (-∞, 0) U (0, ∞)

Range: (-∞, 0) U (0, ∞)

Points: (-1, -1), (1, 1)

Rational function: $f(x) = \frac{1}{x^2}$



Domain: $(-\infty, 0) \cup (0, \infty)$

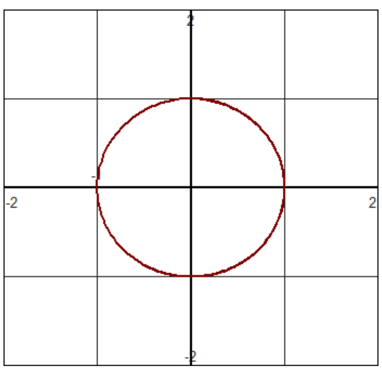
Range: $(0, \infty)$

Points: (-1, 1), (1, 1)

Circles

A circle consists of the set of points that are equidistant from a fixed point. The fixed point is the center of the circle and the distance between the center and every point on the circle is the radius of the circle. The standard equation of a circle centered at the origin is $x^2 + y^2 = r^2$ where the radius of the circle is r.

Unit Circle (circle of radius 1 centered at the origin): $x^2 + y^2 = 1$



Domain: [-1, 1]

Range: [-1,1]

Points: (-1, 0), (1, 0), (0, 1), (0, -1)

Graphs can be used to solve equations and inequalities and estimate values of functions.

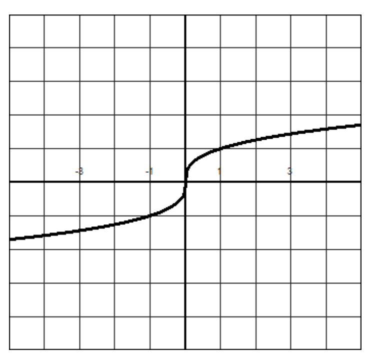
Example:

Refer to the graph of $f(x) = \sqrt[3]{x}$ shown. Estimate answers to one decimal place if necessary.

A. Estimate the value of $\sqrt[3]{2}$.

B. Solve the equation $\sqrt[3]{x} = -1.5$.

C. Solve $\sqrt[3]{x} > 1$.



Solution:

A. To estimate $\sqrt[3]{2}$ from the graph of $\sqrt[3]{x}$ shown, we need to look at the point on the graph where x = 2. Estimating the y-value of the point from the graph gives $\sqrt[3]{2} \approx 1.3$. This is an estimate so your estimate may be slightly different but it should be clear than the value should be more than 1 and less than 1.5 from the graph.

B. To solve the equation $\sqrt[3]{x} = -1.5$ using the graph, the output value should be -1.5. Draw a horizontal line at y = -1.5 on the graph and estimate the x-value of the point on the graph with the y-value of -1.5. This occurs at approximately x = -3 so that is the approximate solution to the equation.

C. To solve the inequality $\sqrt[3]{x} > 1$ using the graph, first solve the equation Solve $\sqrt[3]{x} = 1$. The output of the graph is 1 at x = 1. We are looking for where the output values are greater than 1 or where the graph is above the line y = 1. This occurs for values to the right of x = 1, so the solution to the inequality is (1, ∞).

Symmetry and Even and Odd Functions

Symmetry is often found in art and mathematics. Objects can be symmetric (mirror images) across a vertical or horizontal line or as a rotation around a point.

A graph will be **symmetric to the horizontal axis** (or x-axis) if when the graph is folded along the x-axis, the upper and lower halves match exactly. A function can not be symmetric to the horizontal axis (with the exception of the line y = 0). Also, a graph will be symmetric to the horizontal axis if it contains both the points (x, y) and (x, -y) for every x in its domain.

A graph will be **symmetric to the vertical axis** (or y-axis) if when the graph is folded along the y-axis, the right and left halves match exactly. A function whose graph is symmetric to the y-axis is called an **even function**. Also, a function will be even if f(-x) = f(x) for every x in its domain.

A graph will be **symmetric to the origin** if we spin or rotate the graph about the origin half a turn and the original graph reappears. A function whose graph is symmetric to the origin is called an **odd function**. A function will be odd if f(-x) = - f(x) for every x in its domain.

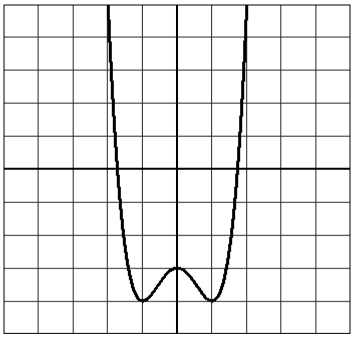
Because this course focuses on functions, we will focus on symmetry about the vertical axis and symmetry about the origin.

Looking at the eight basic functions presented in this section, the absolute value, quadratic and the rational function f(x) = $\frac{1}{x^2}$ are all even functions because they are symmetric to the vertical axis. If folded along the axis, the left and right sides are the same. The odd functions are the linear, cube root, cubic and rational function f(x) = $\frac{1}{x}$ as these are symmetric to the origin. Circles are symmetric to the horizontal and vertical axis and also the origin.

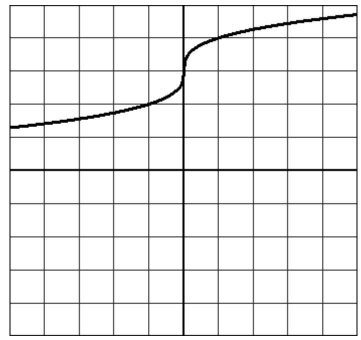
Examples:

Identify whether the following graphs are even, odd, or neither.

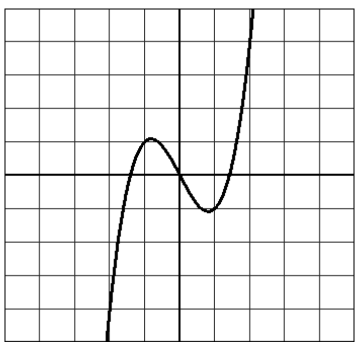
1.



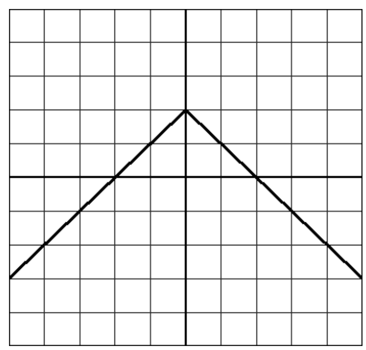
2.



3.



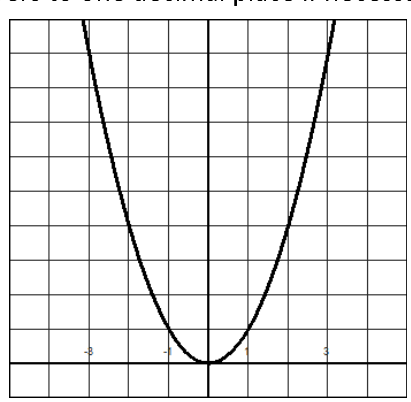
4.



Graphs 1 and 4 are even functions because they are symmetric to the vertical axis. Graph 3 is odd because it is symmetric to the origin. Graph 2 is neither even nor odd because it does not have either type of symmetry.

1.3 Homework:

1. Refer to the graph of f(x) = x^2 shown. Estimate answers to one decimal place if necessary.

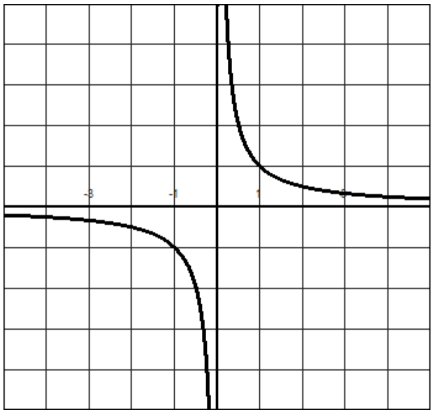


A. Estimate the value of $(2.5)^2$.

B. Find all numbers whose square is 4.

C. Find all solutions to the equation $x^2 = 7$.

2. Refer to the graph of y = $\frac{1}{x}$ shown. Estimate answers to one decimal place if necessary.



A. Estimate the value of $\frac{1}{2.4}$.

B. Find all solutions to the equation $\frac{1}{x} = -3.2$.

C. Solve $\frac{1}{x} \geq 2$.

3. Refer to the graph of f(x) = $0.5x^3$ shown. Estimate answers to one decimal place if necessary.

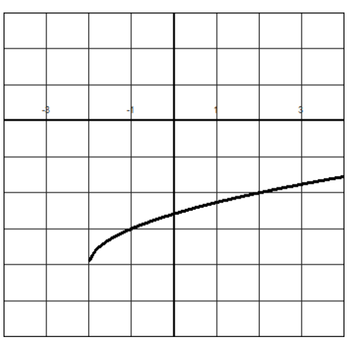
A. Estimate $0.5(-2.5)^3$.

B. Solve the equation $0.5x^3 = 4$.

C. Solve $0.5x^3 \leq 2$.

Each of the following graphs is a variation of one of the eight basic functions. Identify the basic function for each problem.

4.



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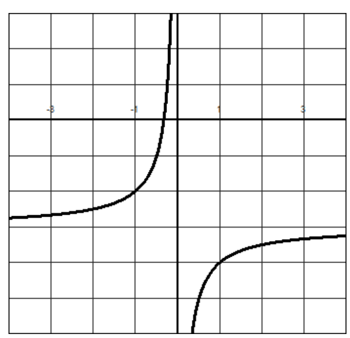

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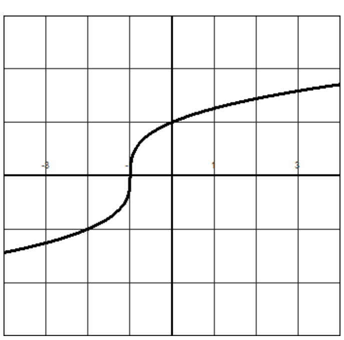

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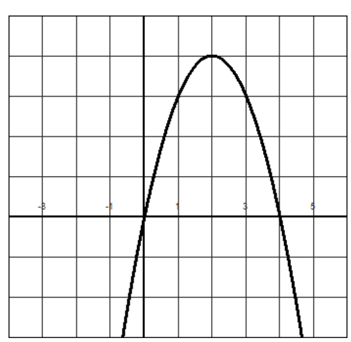

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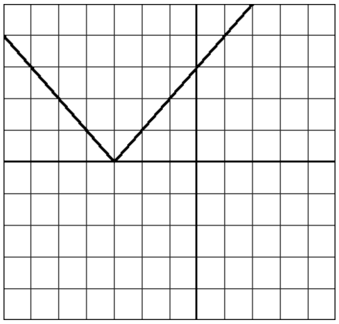

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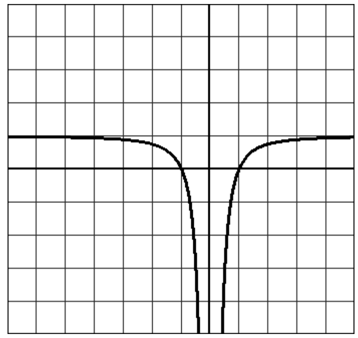
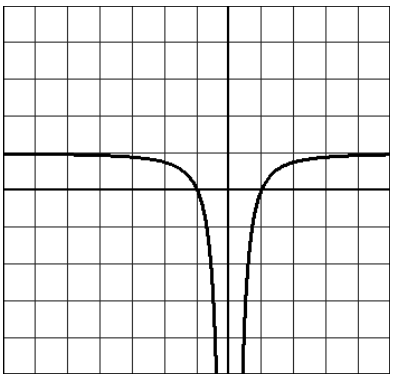


Are the following functions even, odd, or neither?

10.



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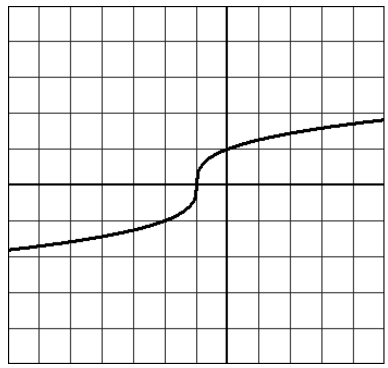

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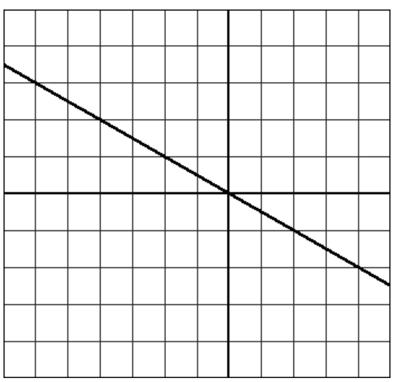
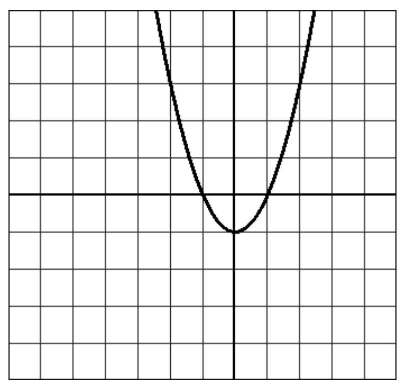
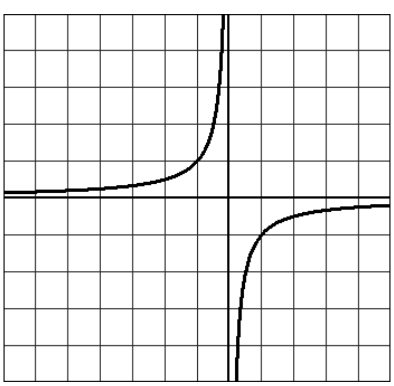


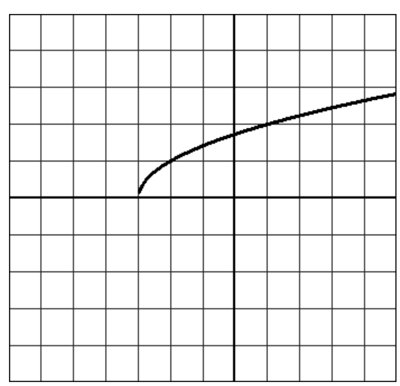
13.



14.



15.



Determine graphically if the following functions are even or odd or neither.

16. y = $|x| + 3$

17. y = $|x - 2|$

18. y = $x^3 + 1$

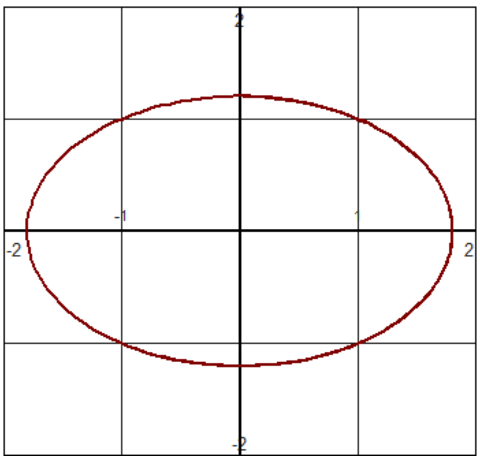
19. y = $2x^3$

20. y = $x^4 - 5x^2 + 4$

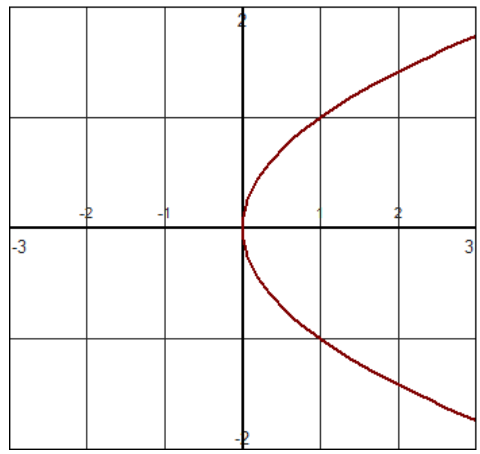
21. y = $x^2 + x$

Identify whether the following graphs are symmetric to the horizontal axis, vertical axis, and/or the origin.

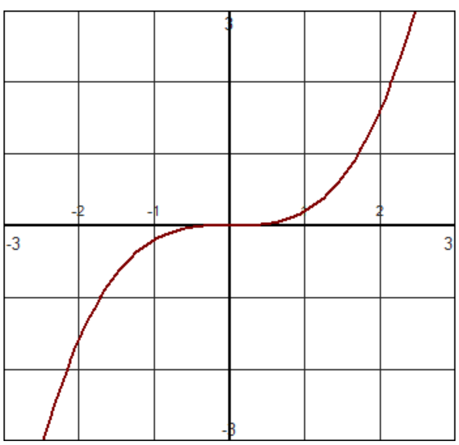
22.



23.



24.



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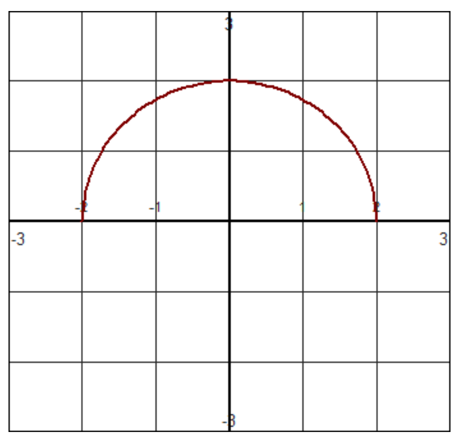


1.4 Tranformations of Graphs

In this section, new graphs are created from the basic shapes learned in the previous section. Variations caused by certain changes in the equations of functions are called transformations. This section examines how changes to the basic equations of functions affect the graph.

Vertical translations:

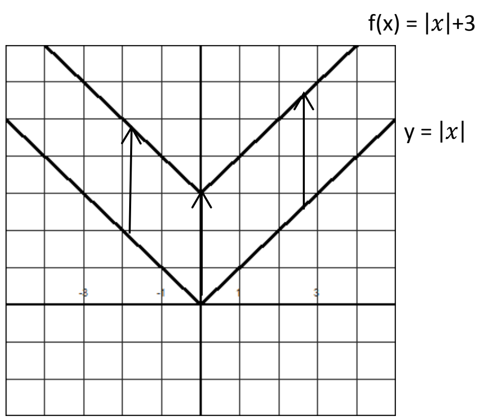
A vertical translation shifts a function upward or downward from its original position.

Compared to the graph of y = f(x), the graph of y = f(x) + k (k > 0) will shift the graph upward k units.
Compared to the graph of y = f(x), the graph of y = f(x) – k (k > 0) will shift the graph downward k units.

If we compare the function y = $|x|$ to f(x) = $|x|$ + 3 by looking at a table of values, the output values have all been increased by 3 when the input values are equal.

x	-2	-1	0	1	2
y = $|x|$	2	1	0	1	2
f(x) = $|x| + 3$	5	4	3	4	5

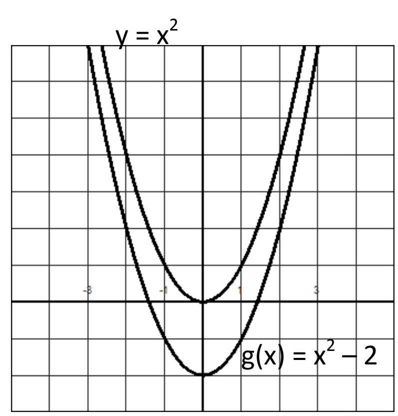
The graphs are shown below. Notice that the entire graph has been shifted up 3 units to form the new translated graph. The new graph is the same shape and size as the original but has just been shifted to a new location.



Examples:

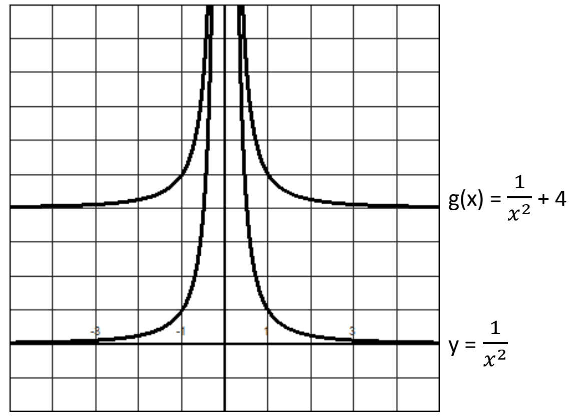
1. Graph g(x) = x^2 – 2.

   The new graph should be shifted down 2 units from the position of the graph of y = x^2 but retain the same shape and size.



2. Graph $g(x) = \frac{1}{x^2} + 4$.

   The new graph will have the same size and shape as $y = \frac{1}{x^2}$ but will be shifted up 4 units.



Horizontal Translations:

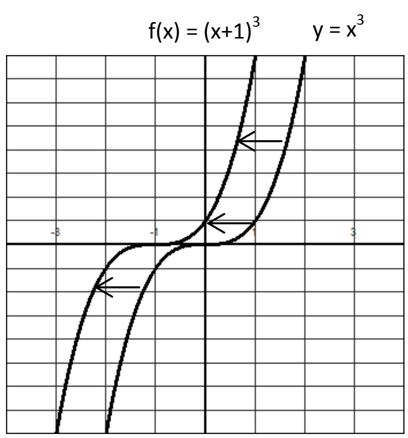
A horizontal translation shifts a function to the left or to the right from its original position.

Compared to the graph of y = f(x), the graph of y = f(x + h) (h > 0) will shift the graph left h units.
Compared to the graph of y = f(x), the graph of y = f(x – h) (h > 0) will shift the graph right h units.

If we compare the function $y = x^3$ to $f(x) = (x + 1)^3$, we must move 1 unit to the left on the table to find the same value output value of f(x) as the output of y.

x	-2	-1	0	1	2
$y = x^3$	-8	-1	0	1	8
$f(x) = (x + 1)^3$	-1	0	1	8	27

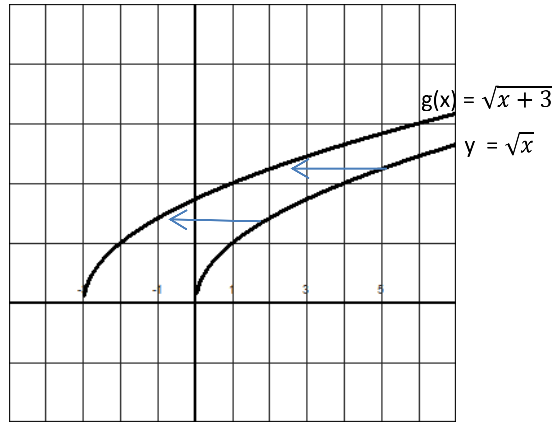
The graphs are shown below. Notice that the entire graph has been shifted left 1 unit to form the new translated graph. The new graph is the same shape and size as the original but has just been shifted to a new location.



Examples:

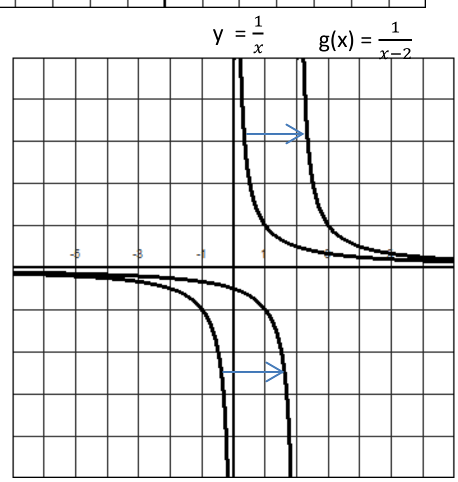
1. Graph g(x) = $\sqrt{x+3}$.

   The new graph should be shifted left 3 units from the position of the graph of y = $\sqrt{x}$ but retain the same shape and size.

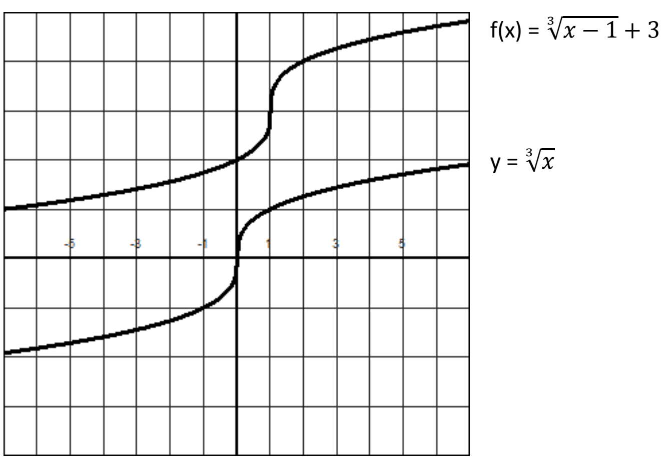


2. Graph g(x) = $\frac{1}{x-2}$

   The new graph should be shifted right 2 units from the position of the graph of y = $\frac{1}{x}$ but retain the same shape and size.



Graphs can have both vertical and horizontal translations. The graph of f(x) = $\sqrt[3]{x-1}+3$ would have the same shape as y = $\sqrt[3]{x}$ but would be shifted both 3 units upward and 1 unit to the right.

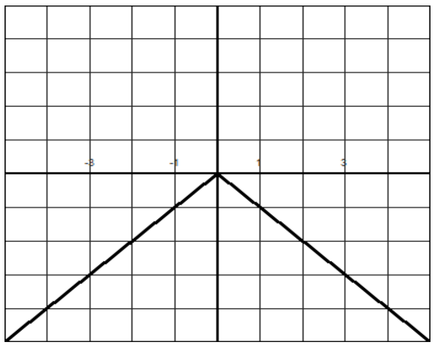


Reflections:

The graph of –f(x) is a reflection of the graph of f(x) across the x-axis.

If we compare y = $|x|$ to f(x) = -$|x|$, then the output values of the two functions are opposites.

x	-2	-1	0	1	2
y = $|x|$	2	1	0	1	2
f(x) = -$|x|$	-2	-1	0	-1	-2



f(x) = - $|x|$

Stretching and Shrinking:

A scale factor stretches or shrinks a graph. The output values on the new graph are multiples of the original output values. This type of transformation changes the size of the graph but not the general shape.

Compared to the graph of y = f(x), the graph of y = af(x) (a > 1) will stretch the graph. For many graphs the result of stretching is that the graph will look narrower than the original.

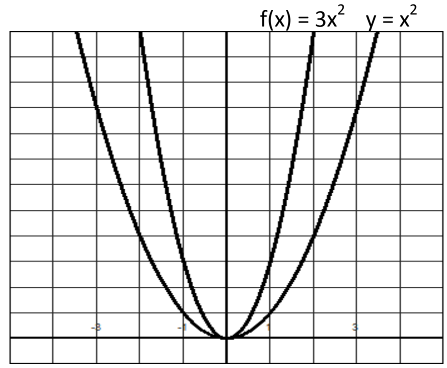
Compared to the graph of y = f(x), the graph of y = af(x) (0 < a < 1) will shrink the graph. For many graphs the result of shrinking is that the graph will look wider than the original.

If we compare the function y = x^2 to f(x) = $3x^2$ by looking at a table of values, the output values have all been multiplied by 3 when the input values are equal.

x	-2	-1	0	1	2
y = x^2	4	1	0	1	4
f(x) = $3x^2$	12	3	0	3	12

The graphs are shown below. Notice that the transformed graph appears narrower than the original. The output values have been "stretched upward'.

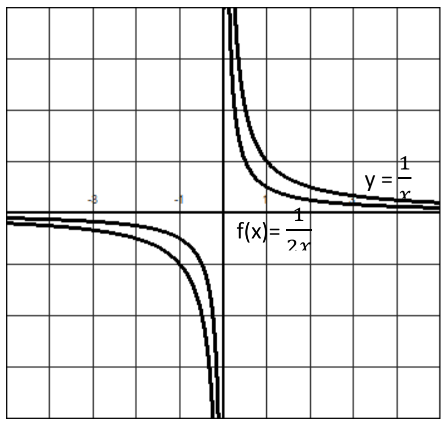
f(x) = $3x^2$ y = x^2



A value of “a” between 0 and 1 will cause the graph to shrink or the y-values to become smaller than the original values. For example, if we compare y = $\frac{1}{x}$ to f(x) = $\frac{1}{2} \cdot \frac{1}{x} = \frac{1}{2x}$, we can see that all the output values on the transformed function are half the size as the original output values.

x	-2	-1	0	1	2
y = $\frac{1}{x}$	-1/2	-1	undefined	1	1/2
f(x) = $\frac{1}{2x}$	-1/4	-1/2	undefined	1/2	1/4

The graphs are shown below. The transformed graph has been “shrunk vertically”. This graph is closer to the horizontal axis.

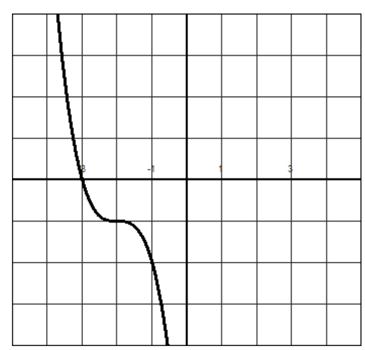


All of the transformations can be performed on any function; not just on the basic shapes. Also, the transformations can be combined on the same function.

Examples:

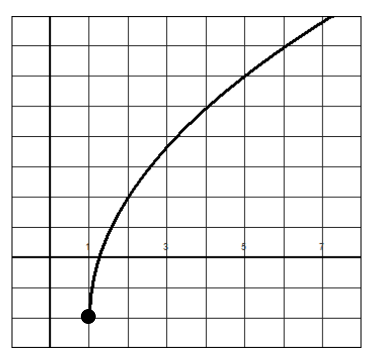
1. Graph f(x) = -(x + 2)^3 – 1.

    This graph will be reflected about the x-axis, shifted to the left 2 units, and shifted down 1 unit.



2. Graph g(x) = 4$\sqrt{x-1} - 2$.

    This graph will have the shape of y = $\sqrt{x}$ but will be stretched vertically by 4, shifted to the right 1 unit and shifted down 2 units.



Circles

Recall that the standard equation of a circle centered at the origin with a radius of r is $x^2 + y^2 = r^2$. A circle with an equation in the form $(x - h)^2 + (y - k)^2 = r^2$ has a center of (h, k) and the radius of the circle is r. When we shifted the basic shapes, the number inside the parentheses with the variable x shifted the graph to the right or left and the same is true for circles. In addition, the number inside the parentheses with the variable y shifts the graph up or down.

Examples:

1. Find the center and radius of the circle with the equation $(x - 1)^2 + (y + 5)^2 = 49$.

   The center is (1, -5) and the radius is 7.

2. Find the center and radius of the circle with the equation $x^2 + (y - 3)^2 = 9$.

   The center is (0, 3) and the radius is 3.

3. Find the equation of a circle of radius 4 with a center at the origin.

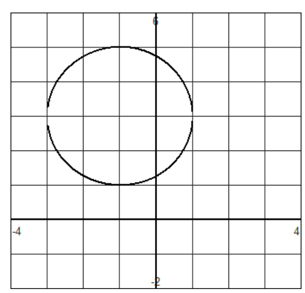
   The equation is $x^2 + y^2 = 16$.

4. Find the equation of the circle shown.

   The center of the circle is (-1, 3) with a radius of 2.

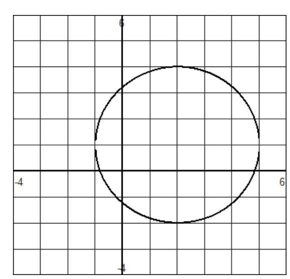
   Therefore, the equation of the circle is

   $(x + 1)^2 + (y - 3)^2 = 4$.



5. Sketch a graph of the circle with equation $(x - 2)^2 + (y - 1)^2 = 9$.

   From the equation, the center is (2, 1) and the radius is 3.



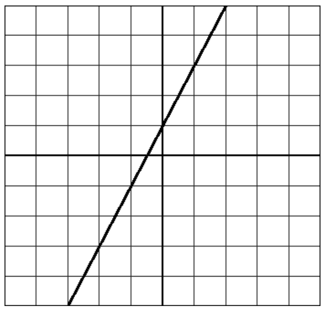
1.4 Homework:

Write in words what the symbols mean when you translate the graph of f(x).

1. f(x + 3)
2. f(x) – 5
3. f(x – 1) – 7
4. –f(x) + 4
5. 2f(x + 6)
6. ½ f(x) – 2
7. -3f(x – 5) + 1
8. 0.3f(x + 4) + 8

Write the equation of each graph. Assume the scales on both axes are 1 unit.

9.



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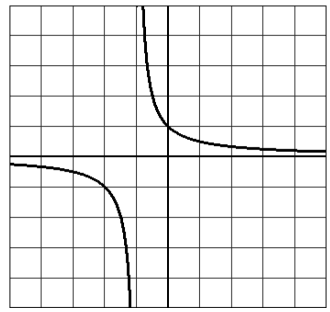

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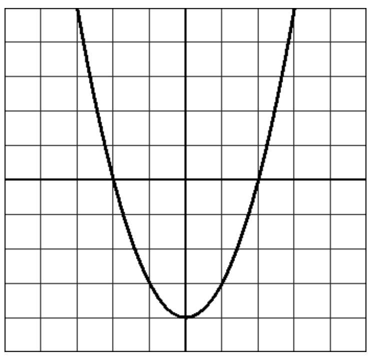

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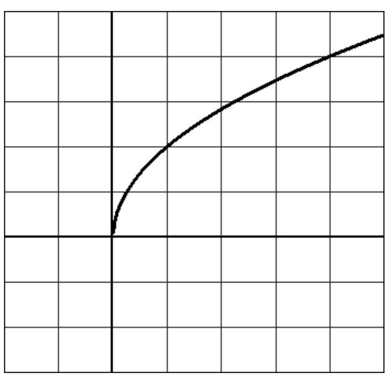

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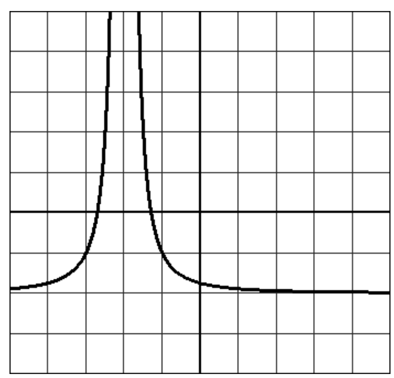

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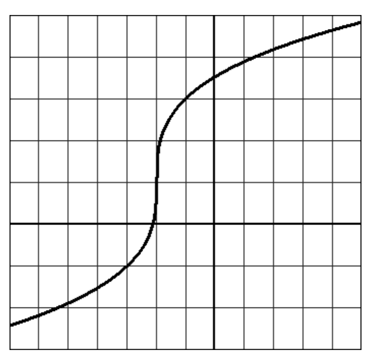


Sketch graphs of the following functions.

17. f(x) = $(x + 3)^2 - 1$

18. G(x) = $\sqrt[3]{x-1}$

19. h(x) = $\sqrt{x+1} - 2$

20. F(x) = $\frac{2}{x^2}$

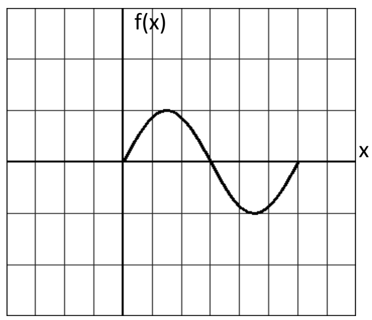
21. g(x) = $\frac{1}{x} + 3$

22. f(x) = -3x + 2

23. f(x) = $2x^3 + 1$

24. H(x) = $-|x-2| - 4$

The graph of a function, f(x), is shown. Describe each transformation of the graph and give a formula in terms of the original function.



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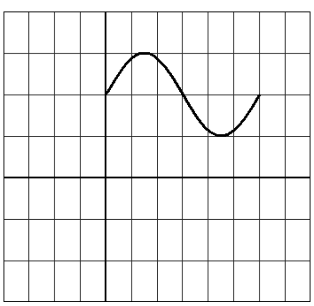

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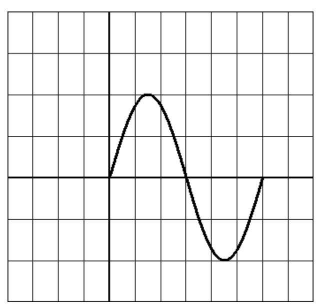

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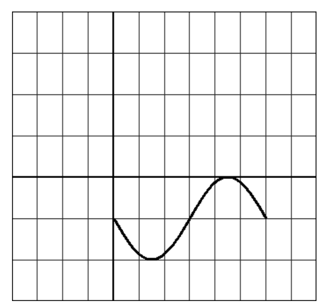


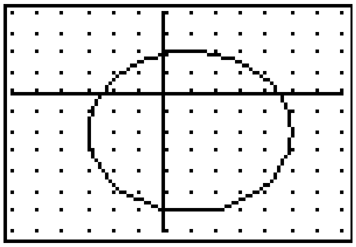
28. Find the center and radius of the circle.

A. $x^2 + y^2 = 25$ B. $x^2 + y^2 = 70$

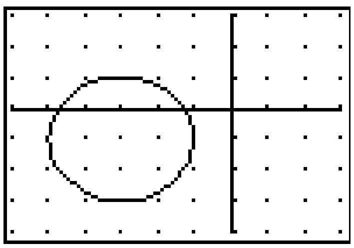
29. Find the center and radius of the circle and sketch a graph.

A. $(x - 3)^2 + y^2 = 36$ B. $(x + 4)^2 + (y - 2)^2 = 100$

30. Use the graph to write the equation of the circle.



31. Use the graph to write the equation of the circle.



32. Find the equation of a circle with radius 5 and center at (9, 2).

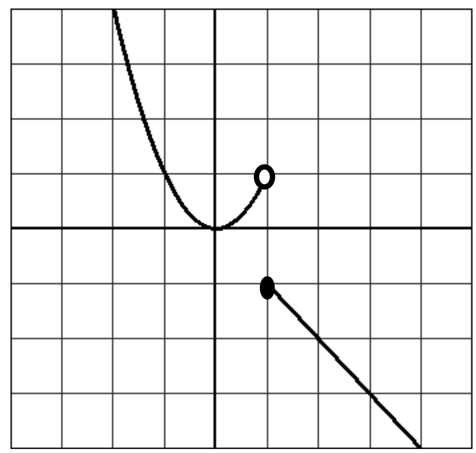
33. Find the equation of a circle with radius 9 and center at (-3, -5).

1.5 Piecewise Functions

A piecewise function is a function that is defined by different formulas on different intervals on the x-axis. To graph, we consider each interval separately and graph the formula for that interval. The function is the graph of all the individual pieces together on one set of axes. If the pieces do not connect, then we use open circles and closed circles to indicate where the point is for the value of x where the graph is discontinuous.

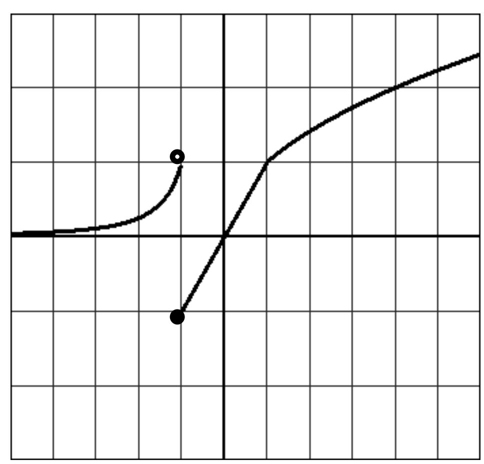
Examples:

1. Graph the function f(x) = $\begin{cases} x^2, & x < 1 \\ -x, & x \geq 1 \end{cases}$

   We can think of the coordinate system as being divided into two sections; one section is to the left of the line x = 1 and the other section is to the right of the line x = 1. Some people find it easier to graph the formula on the entire coordinate plane and then erase the part that is not on the interval where the formula is defined. First, graph f(x) = x^2 on the portion of the coordinate plane to the left of x = 1. The endpoint where x = 1 on this portion of the graph will be an open circle because the interval is x < 1. Then, graph f(x) = -x on the portion of the plane to the right of x = 1. The endpoint of this portion of the graph will have a solid circle because the interval is x ≥ 1.

2. Graph the function g(x) = $\begin{cases} \frac{1}{x^2}, & x < -1 \\ x, & -1 \leq x \leq 1 \\ \sqrt{x}, & x > 1 \end{cases}$

   In this example, the coordinate system will be divided into three parts; one part to the left of x = -1, the part between x = -1 and x = 1 and the part to the right of x =1. The function f(x) = $\frac{1}{x^2}$ will be graphed for x < -1, the graph of f(x) = x will be graphed for values of x between -1 and 1, and finally the graph of f(x) = $\sqrt{x}$ will be graphed to the right of x = 1. Looking at the graph, the first two sections of this piecewise function do not intersect; checking the inequality signs leads to an open circle at x = -1 on the graph of f(x) = $\frac{1}{x^2}$ and a closed circle at x = -1 on the graph of f(x) = x. At x = 1, the two pieces of the graphs connect at the point (1, 1) so nothing further is needed.

A function that is constant on each interval of its domain is called a **Step function**. The graph of a step function resembles a staircase.

1.5 Homework:

Sketch the following.

1. f(x) = $\begin{cases} -\frac{1}{2}x+1, & x<0 \\ x^2+1, & x\geq 0 \end{cases}$

2. $g(x) = \begin{cases} x^3, & x<1 \\ \frac{1}{x}, & x\geq 1 \end{cases}$

3. h(x) = $\begin{cases} -2|x|, & x\leq -1 \\ x^2-1, & x>-1 \end{cases}$

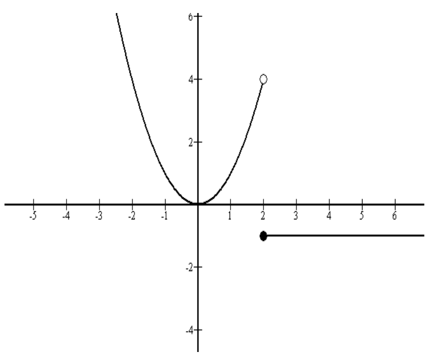
4. F(x) = $\begin{cases} \sqrt[3]{x+1}, & x\leq 0 \\ x^3-1, & x>0 \end{cases}$

5. $G(x) = \begin{cases} -3, & x<-2 \\ 2x+1, & -2\leq x\leq 1 \\ |x-1|+2, & x>1 \end{cases}$

6. H(x) = $\begin{cases} \frac{1}{x^2}, & x<0 \\ -x^2, & 0\leq x<2 \\ x-2, & x\geq 2 \end{cases}$

Write the functions for the following piecewise graphs.

7.



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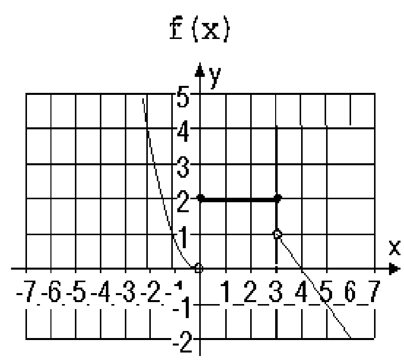

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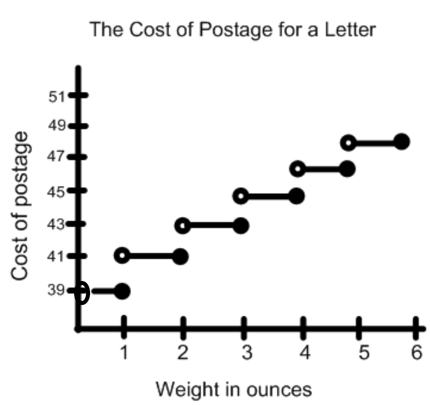

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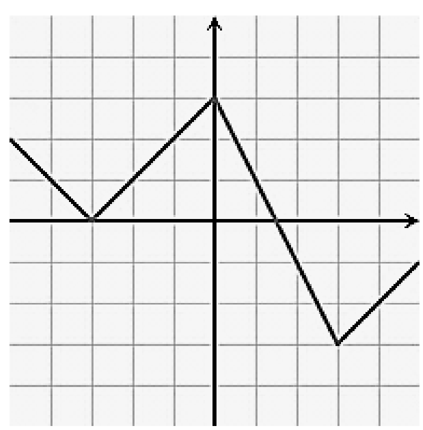


11. Write the function and sketch the piecewise function for the shipping chart shown below. Identify the domain and range.

Shipping Chart

Order Amount	Shipping & Handling
$1.00 to $24.99	**$7.50**
$25.00 to $49.99	**$10.00**
$50.00 to $74.99	**$17.00**
$75.00 to $99.99	**$22.00**
$100.00 +	***FREE!***

12. Capitol One calculates the minimum payment due on a credit card each month as the greater of $15 or 3% of the balance. Write a piecewise function for the minimum payment due. Sketch a graph of the minimum payment due.

1.6 Modeling with Graphs and Average Rates of Change

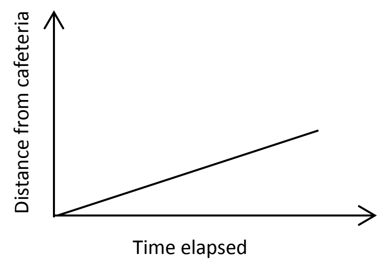
Creating a good model for a situation starts with deciding what kind of function to use. An appropriate model will depend on the general shape of the graph. Should the model be increasing, decreasing or a combination of both? Should the model have a constant slope or is it changing? Is the model a continuous graph or will it be a piecewise graph? In this section, we focus on appropriate models for different scenarios.

Example:

There are three students walking to class from the cafeteria. For each of the following situations, sketch a possible graph of the student's distance from the cafeteria as a function of time.

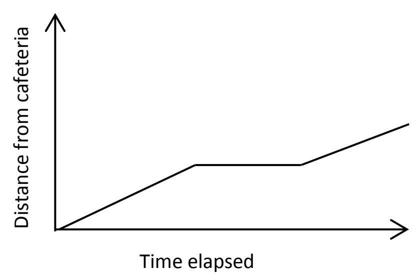
A. Student 1 walks at a steady pace to class.

The graph begins at the origin since the student starts in the cafeteria. His distance will increase as he gets closer to class (further from the cafeteria). Notice that the graph is linear because his pace (rate of change in distance) is steady.



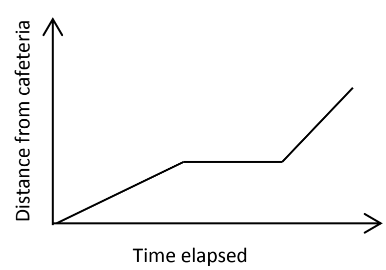
B. Student 2 starts to walk to class at a steady pace but stops part way to class to talk to a friend. Then he continues to class at his original pace.

The student again starts in the cafeteria but while he is talking to his friend his distance from the cafeteria remains constant (horizontal line). As he continues to class, the slope of the last segment should be the same as the slope of the first segment since he is walking at the same pace.



C. Student 3 starts to walk to class at a steady pace, stops part way to talk to a friend. Then, since he is late he hurries the rest of the way to class.

The student again starts in the cafeteria and then while he is talking to his friend his distance from the cafeteria remains constant (horizontal line). As he continues to class, the slope of the last segment should be steeper than the slope of the first segment since he is walking at a faster pace.



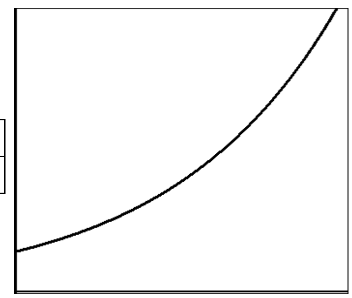
Graphical models do not have to have a constant slope. Various models may be increasing but may increase in different ways. The same is true for models which decrease. The example below shows two increasing models. One model increases at an increasing rate and the other increases but at a decreasing rate.

Example:

A. The number of cases of the flu reported during the beginning of flu season at a local health clinic is an increasing function of time. The number of flu cases is growing at a faster and faster rate.

A table of the number of flu cases reported shows that as time increases, the change in the number of cases is increasing during each time period.

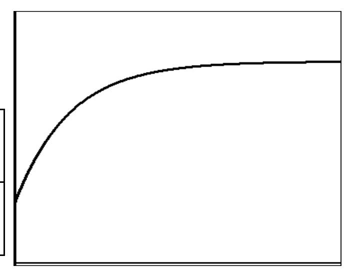
T, days	0	2	4	6	8
C, cases	10	23	51	114	256



B. The temperature of a pie which is placed into a preheated oven increases rapidly at first, then more slowly as it approaches the temperature of the oven.

A table of the temperature of the pie shows a rapid increase in temperature then a much slower increase in temperature.

m, minutes	0	3	6	9	12	15
T, temp in °F	40	180	256	298	322	334

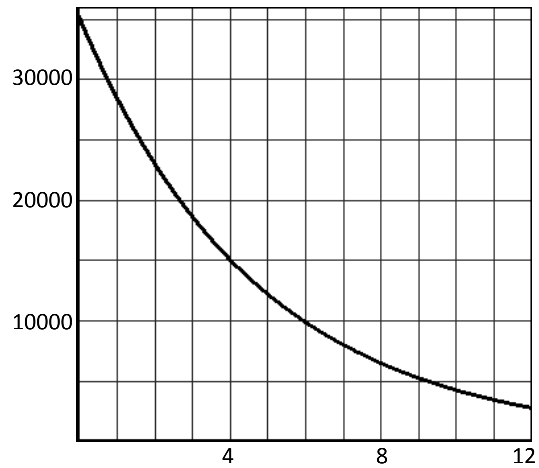


If the slope of a function is not constant, then the rate of change is not constant. In these instances, we can use the average rate of change over a certain interval to describe how one variable is changing in relation to the other. The **average rate of change** is the change in the output values divided by the change in the input values for two distinct points on the graph. Another way to think of this is the average rate of change is the slope of the line segment which connects the two given points (this line is called a secant line). For example, in the example above, the average rate of change of the temperature of the pie between 0 and 3 minutes is $\frac{180-40}{3-0} = 46.7$ °F/min. As the temperature is rising more slowly, the average rate of change over other intervals should be smaller. The average rate of change from 9 to 15 minutes is $\frac{334-298}{15-9} = 6$°F/min.

Example:

Sue buys a car for \$35000. The table and graph show the value of her vehicle over time. The value of a car decreases most rapidly during the first year of ownership and then decreases at a slower rate each year.

Time, years	Value, \$
0	35000
1	28350
2	22964
3	18600
4	15066
5	12200
6	9885
7	8007



A. Find the average rate of change of the value of the car during the first year.

Average rate of change = $\frac{28350-35000}{1-0} = -6650$

This means that the car's value decreased an average of \$6650 per year during the first year she owned it.

B. Find the average rate of change of the value of the car between years 4 and 6.

Average rate of change = $\frac{9885-15066}{6-4} = -2590.50$

This means that the car's value decreased an average of \$2590.50 per year between years 4 and 6 of ownership.

Example:

An object is dropped off the roof of a 200-foot tall building. The object's height after t seconds is given by $h(t) = 200 - 16t^2$, where h is measured in feet.

A. Find the average rate of change from 0 to 1 second.

Using the equation, we find that the object is 200 feet high at 0 seconds and 184 feet high at 1 second. The average rate of change is:

$\frac{184-200}{1-0} = -16$ feet per second

The object is falling at an average rate of 16 feet per second during the first second after it is dropped.

B. Find the average rate of change from 2 to 3 seconds.

Using the equation, we find that the object is 136 feet high at 2 seconds and 56 feet high at 3 seconds. The average rate of change is:

$$\frac{56-136}{3-2} = -80 \text{ feet per second}$$

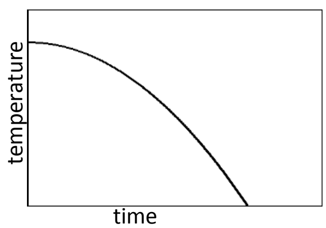
The object is falling at an average rate of 80 feet per second during the time from 2 to 3 seconds after the object is dropped.

1.6 Homework:

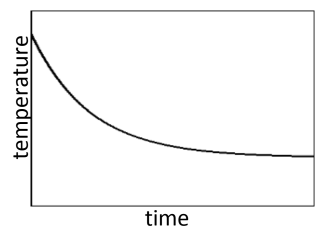
Which of the following graphs best represents the scenario?

1. A cup of coffee cooled off rapidly at first, and then gradually approached room temperature.

A.

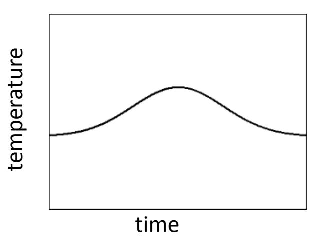


B.

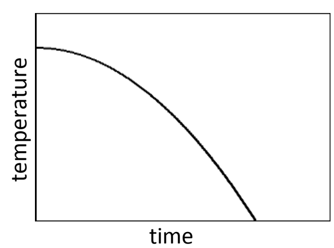


2. The temperature of a person during an illness.

A.

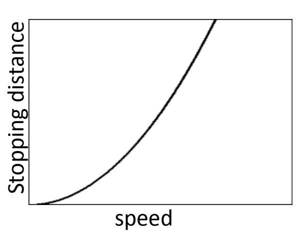


B.

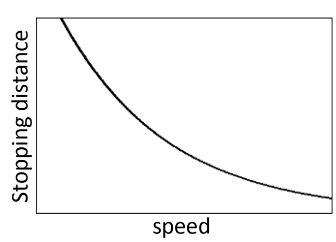


3. The stopping distance for a car traveling various speeds

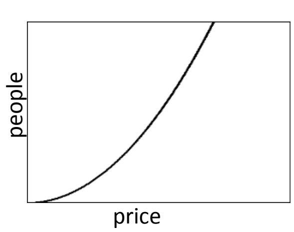
A.

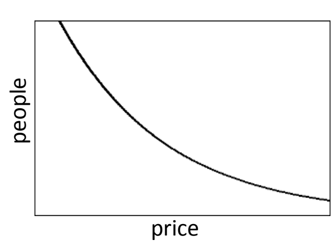


B.



4. The number of people willing to buy a new computer as a function of its price.





A. B.

5. Match the scenario to the appropriate table of values.

A. A cup of coffee cools off rapidly at first, then gradually approached room temperature.

B. The number of bacteria in a person during the course of an illness is a function of time. It increases rapidly at first, then decreases slowly as the patient recovers.

C. The price of an item rose slowly at first, but due to high demand is now rising rapidly.

D. A student drops their algebra book off the top of a building. The height of the book is a function of time, decreasing ever more rapidly as gravity accelerates its descent.

I.

x	1	2	3	4	5	6	7
y	20	560	350	230	150	90	40

II.

x	1	2	3	4	5	6	7
y	190	135	100	87	80	77	75

III.

x	0	1	2	3	4	5	6
y	160	155	140	116	82	38	0

IV.

x	1	2	3	4	5	6	7
y	10	12	15	27	50	90	175

6. Match each graph with the function it illustrates.

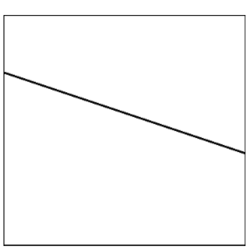
A. Unemployment was falling but is now steady.

B. The birthrate rose steadily but it is now beginning to fall.

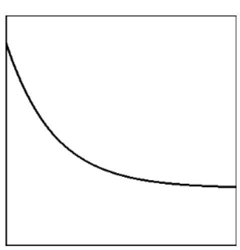
C. The price of gasoline has fallen steadily over the last few months.

D. Inflation, which was rising steadily, is now rising more rapidly.

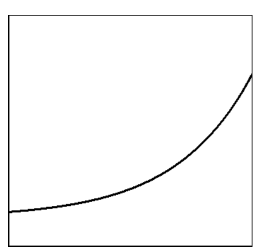
I.



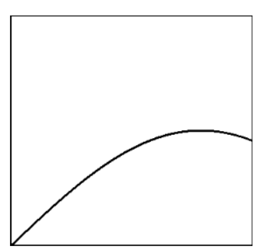
II.



III.



IV.



Sketch possible graphs to illustrate the situations described below.

7. Cathy leaves school to drive home. On the way, she realizes that she forgot to pick up a worksheet so she turns around and goes back to school to get it. Then, halfway home, she stops for gas before continuing to her house. Graph the distance between Cathy and her home as a function of time.

8. The sales tax on a purchase, as a function of its price.

9. The height of a person as a function of their age, from birth to adulthood.

10. The height above the ground of a ball dropped from 6 feet.

11. When learning a new language, the number of words you learn increases slowly at first, then increases more rapidly, and finally starts to level off.

12. A person's income in terms of the hours worked if they get paid a flat amount per hour.

13. Jack drives from home to meet his friend at the gym which is halfway between their houses. After they workout, they go to the friend's house to watch some TV. Later, Jack drives back home. Graph the distance between Jack and his house as a function of time.

14. The level of the tide at the beach rises and then falls over each 12 hour time period.

Find the average rate of change for the following:

15. $f(x) = -3x + 7$ from $x = -1$ to $x = 4$

16. $s(t) = \sqrt{4t - 1}$ from $t = 5$ to $t = 17$

17. $g(x) = 2x^3 + 8x - 3$ from $x = -2$ to $x = 4$

18. $E(x) = 3x^2 - 5$ from $x = -2$ to $x = 3$

19. $h(x) = \dfrac{x+1}{x^2 + x - 6}$ from $x = -2$ to $x = 1$

20. $L(x) = |3x + 5|$ from $x = -3$ to $x = 3$

21. The Fahrenheit temperature at a pig farm on a warm spring evening x hours past noon is given by

$T(x) = 0.27x^2 - 7.23x + 124.25$. Find the average rate of change of T from 7 pm to 11pm. Interpret your answer.

22. A spherical balloon is being inflated. The surface area of a sphere is $S = 4\pi r^2$.

A. Find the average rate of change of the surface area with respect to the radius r when r changes from 2 inches to 3 inches.

B. Find the average rate of change when the radius changes from 3 inches to 5 inches.

23. A particle is moving along the curve $s(t) = t^2 - 6t + 9$ where t is in seconds and s in feet. Find the average velocity from t = 1 to t = 2 seconds.

24. As reported in the Orlando Sentinel on August 23, 1998, the attendance at Disney World parks in millions is shown in the table. Find the **average rate of change** of admissions to the Disney World parks between the following years:

A. 1980 to 1983

B. 1987 to 1988

C. 1992 to 1997

year	Admissions (millions)	year	Admissions (millions)
1980	13.8	1990	32.8
1982	12.6	1991	28.2
1983	22.7	1992	29.6
1985	21.9	1993	29.1
1986	23.9	1994	27.6
1987	27.0	1996	34.4
1988	25.5	1997	36.3

1.7 Variation

Two types of functions are commonly used in modeling. These models are direct variation and inverse variation.

Direct variation occurs when the ratio of the two variables remains constant. In simpler terms, that means if A is always twice as much as B, then they directly vary. If a pound of apples costs $2.59, and I buy 1 pound, the total cost is $2.59. If I buy 2 pounds, the price is $5.18. In this example the total cost of the apples and the number of pounds purchased are subject to direct variation -- the ratio of the cost to the number of pounds is always 2.59.

Definition: y **varies directly** as x if y = kx where k is a positive constant called the **constant of variation**.

Varies directly means the same as **directly proportional**. Note that for direct variation when one variable is zero, the other variable is zero. Therefore, all graphs should start at the origin. Also, as one variable increases, so does the other variable so the graph should be an increasing function.

Examples:

1. If y varies directly as x, and y = 8 when x = 12, find k and write an equation that expresses this variation.

    Since y varies directly as x, the equation will have the form y = kx. Plug the given values into the equation for x and y.
    8 = k(12)
    Divide both sides by 12 to find k: 8/12 = k
    2/3 = k
    Go back to y = kx and replace k with 2/3.
    y = (2/3)x

2. The number of cups of liquid varies directly as the number of quarts of liquid and there are 12 cups of liquid in 3 quarts. Find k and write an equation that expresses the relation between cups and quarts.

    Since cups vary directly as quarts, the equation will have the form c = kq. Plugging in the given values for c and q gives:
    12 = k(3)
    Divide both sides by 3 to find k: 12/3 = k
    4 = k
    Go back to c=kq and replace k with 4.
    c = 4q.

The idea of direct variation can be generalized to include situations where the output variable is proportional to a power of the input variable. In this case, **y varies directly as a power of x** if $y = kx^n$ where k and n are positive constants. The constant of variation is k.

Example:

The area of a circle varies directly as the square of its radius. A circle of radius 4 inches has an area of 50.265 square inches.

A. Express the area of the circle, A, as a function of the length of the radius, r.

Since the area varies directly as the square of the radius, the equation will have the form $A = kr^2$. Plugging in the given values for A and r:

$50.265 = k(4)^2$

$50.265 = 16k$

$3.1416 = k$

Go back and replace k in the equation.

$A = 3.1416r^2$

B. Find the area of a circle of radius 11 inches.

Using the equation found in part A, substitute r = 11 and solve for A.

$A = 3.1416(11)^2$

A = 380.1336 square inches

Inverse variation:

Inverse variation describes a decreasing function. As one variable increases, the other variable decreases. For instance, a biker traveling at 8 mph can cover 8 miles in 1 hour. If the biker's *speed decreases* to 4 mph, it will take the biker 2 hours (*an increase of one hour)*, to cover the same distance.

Definition: y **varies inversely** as x if $y = \frac{k}{x}$, x ≠ o, where k is the constant of variation.

Note that as x increases, y will decrease so the graph will be a decreasing function. Also, the function is undefined when x = 0 and y will never be zero.

Examples:

1. If y varies inversely as x and when x = 3, y = 8, find k and write an equation that expresses this variation.

   Since y varies inversely as x, the equation will have the form $y = \frac{k}{x}$. Plug the given values into the equation for x and y.

   $8 = \frac{k}{3}$

   Multiply both sides by 3 to find k: k = 24

   Go back to $y = \frac{k}{x}$ and replace k with 24.

   $y = \frac{24}{x}$

2. The number of potential buyers of a house decreases as the price of the house increases. In a particular city, houses priced at $100 thousand dollars will have 1000 potential buyers.

   A. If the number of potential buyers of a house is inversely proportional to the price of the house, find an equation that describes the demand for houses as it relates to price.

      Since the number of buyers, B, varies inversely as the price of the house, p, the equation will have the form B = $\frac{k}{p}$. Plug the given values into the equation for p and B.

      1000 = $\frac{k}{100}$

      Multiply both sides by 100 to find k: k = 100,000

      Go back to B = $\frac{k}{p}$ and replace k with 100,000.

      B = $\frac{100000}{p}$

   B. How many potential buyers will there be for a 2 million dollar house?

      The equation is in terms of p in thousands of dollars so 2 million dollars is p = 2000 (thousands of dollars). Substituting into the equation found in part A, we get:

      B = $\frac{100000}{2000} = 50$

      So, a house that is priced at 2 million dollars will have 50 potential buyers.

The idea of inverse variation can be generalized to include situations where the output variable is inversely proportional to a power of the input variable. In this case, **y varies inversely as a power of x** if y = $\frac{k}{x^n}$ where k and n are positive constants. The constant of variation is k.

Example:

The intensity of radio waves varies inversely as the square of the distance from its source. A radio station broadcasts a signal that is measured at 0.016 watt per square meter by a receiver that is 1 kilometer away.

A. Write an equation that gives the signal strength, S, as a function of distance, d.

   Since the signal strength, S, varies inversely as the square of the distance the equation will have the form S = $\frac{k}{d^2}$. Plug the given values into the equation for d and S.

   0.016 = $\frac{k}{1^2}$

   k = 0.016

   Go back to S = $\frac{k}{d^2}$. and replace k with 0.016.

   S = $\frac{0.016}{d^2}$.

B. If you live 8 kilometers from the station, what is the signal strength you will receive?

Substituting into the equation found in part A:
S = $\frac{0.016}{8^2}$ = 0.00025 watt per square meter

Joint and Combined Variation

Joint variation occurs when the values of one variable vary directly based upon the values of two or more variables.

Combined variation describes a situation where a variable depends on two (or more) other variables, and varies directly with some of them and varies inversely with others (when the rest of the variables are held constant).

Examples:

1. If y varies jointly as x and z, and y = 33 when x = 9 and z = 12, find y when x = 16 and z = 22.

   The equation will have the form y = kxz since y varies jointly as both x and z. Substituting into the equation to solve for k gives 33 = k (9)(12). Thus, 33 = 108k and k = $\frac{11}{36}$.
   The general equation is y = $\frac{11}{36}$ xz. To find the value of y when x = 16 and z = 22, plug into this equation. y = $\frac{11}{36}$ $(16)(22)$ = $\frac{968}{9}$ ≈ 107.56.

2. For a given interest rate, simple interest varies jointly as principal and time. If $2000 left in an account for 4 years earns interest of $320, how much interest would be earned in if you deposit $5000 for 7 years?

   The equation will have the form A = kPT where A is the amount of interest. Plugging in the given values and solving for k:
   320 = k(2000)(4)
   k = 0.04
   If you deposit $5000 for 7 years, then the amount of interest A is
   A = 0.04(5000)(7) = $1400.

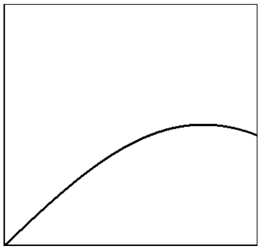
3. If y varies directly as x and inversely as z, and y = 24 when x = 48 and z = 4, find x when y = 44 and z = 6.

   The equation will have the form y = $\frac{kx}{z}$. Solving for k gives:
   24 = $\frac{k(48)}{4}$
   k = 2
   Solving for x when y = 44 and z = 6:
   44 = $\frac{2x}{6}$
   x = 132

1.7 Homework:

Can the graphs shown represent direct variation, inverse variation, or neither?

1.



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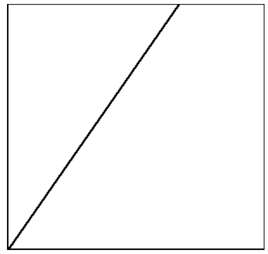

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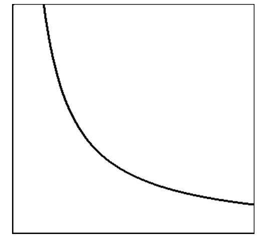

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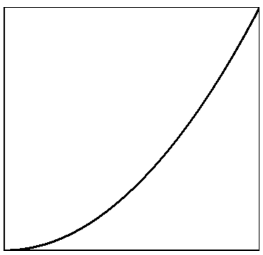

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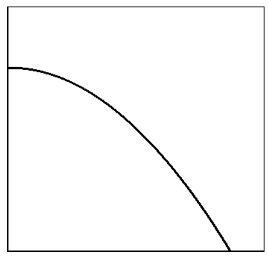
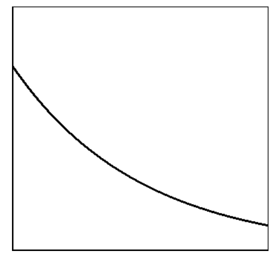


6.



Could the following represent direct variation, inverse variation, or neither? **Explain** your answer.

7.

x	y
0	undefined
1	3.5
2	1.75
3.5	1
4.2	0.833

8.

x	y
0	0
1	0.7
2	2.8
4	11.2
7	34.3

9.

x	y
0	1
1	5
2	9
3.5	15
4.2	17.8

10.

x	y
0	7
1	6.5
2	6.75
3	6.875
4	6.9375

11. Use the values in the table to find the constant of variation, k, and write y as a function of x. Then fill in the rest of the table with the correct values.

y varies inversely with the square of x.

x	y
2	
5	6
	1.5

12. Use the values in the table to find the constant of variation, k, and write y as a function of x. Then fill in the rest of the table with the correct values.

y varies directly with the square root of x.

x	y
1	
4	6
	10

13. Use the values in the table to find the constant of variation, k, and write y as a function of x. Then fill in the rest of the table with the correct values.

y varies inversely with x.

x	y
2	
4	7
	2.5

14. Use the values in the table to find the constant of variation, k, and write y as a function of x. Then fill in the rest of the table with the correct values.

y varies directly with the cube of x.

x	y
2	
4	48
	750

15. The amount of sales tax on a new car varies directly as the purchase price of the car. If a $25000 car pays $1750 in sales tax, express the amount of sales tax as a function of the purchase price. What is the purchase price of a new car which has sales tax of $3500?

16. The volume of a bag of rice varies directly with the weight of the bag. A 2-pound bag contains 3.5 cups of rice. Express the volume, V, of a bag of rice as a function of its weight. How many cups of rice are in a 7-pound bag?

17. In the U.S., the cost of electricity is directly proportional to the number of kilowatts per hour (kWh) used. If a household in Tennessee on average used 3098 kWh per month and had an average monthly electric bill of $179.99, find an equation that gives the cost of electricity in Tennessee in terms of the number of kilowatts per hour used.

18. A brother and sister both have weight (pounds) that varies directly as the cube of height (feet) and they share the same proportionality constant. The sister is 6' tall and weighs 170 pounds. Her brother is 6'4", how much does he weigh?

19. The cost of a house in Florida is proportional (varies directly) to the size of the house. A 2850-square-foot house cost $182,400. Express the cost of the house in terms of the size of the house. What is the cost of a 3640-square-foot house?

20. The number of hours, *h*, it takes for a block of ice to melt varies inversely as the temperature, *t*. If it takes 2 hours for a square inch of ice to melt at 65°, find the constant of proportionality. Write an equation that gives the number of hours it takes for a block of ice to melt in terms of the temperature.

21. In kick boxing, it is found that the force, *f*, needed to break a board, varies inversely with the length, *l*, of the board. If it takes 5 lbs of pressure to break a board 2 feet long, how many pounds of pressure will it take to break a board that is 6 feet long?

22. If an object is dropped off the rim of the Grand Canyon, its speed, v, varies directly with the time, t, that the object has been falling. A rock dropped off the edge of the canyon is falling at a speed of 39.2 meters per second when it passes a lizard on a ledge 4 seconds later. Express the speed, v, as a function of t. What is the speed of the rock after it has fallen for 8 seconds?

23. If f varies jointly as g and the cube of h, and f = 1800 when g = 5 and h = 4, find f when g = 4 and h = 6.

24. Wind resistance varies jointly as an object's surface area and velocity. If an object traveling at 40 mile per hour with a surface area of 25 square feet experiences a wind resistance of 225 Newtons, how fast must a car with 40 square feet of surface area travel in order to experience a wind resistance of 270 Newtons?

25. The volume of a pyramid varies jointly as its height and the area of its base. A pyramid with a height of 12 feet and a base with area of 23 square feet has a volume of 92 cubic feet. Find the volume of a pyramid with a height of 17 feet and a base with an area of 27 square feet.

26. Kinetic energy varies jointly as the mass and the square of the velocity. A mass of 8 grams and a velocity of 6 centimeters per second has a kinetic energy of 110 ergs. Find the kinetic energy for a mass of 5 grams and a velocity of 9 centimeters per second.

27. If a varies jointly as b and c and inversely as the square of d, and a = 120 when b = 5, c = 2, and d = 9, find a when b = 12, c = 9 and d = 9.

28. The volume of gas varies directly as the temperature and inversely as the pressure. If the volume is 230 cubic centimeters when the temperature is 300°K and the pressure is 20 pounds per square centimeter, what is the volume when the temperature is 270°K and the pressure is 30 pounds per square centimeter?

29. The maximum load that a cylindrical column with a circular cross section can hold varies directly as the fourth power of the diameter and inversely as the square of the height. A 9 meter column 2 meters in diameter will support 64 metric tons. How many metric tons can be supported by a column 9 meters high and 3 meters in diameter?

30. The centrifugal force of an object moving in a circle varies jointly with the radius of the circular path and the mass of the object and inversely as the square of the time it takes to move about one full circle. A 6 gram object moving in a circle with a radius of 75 centimeters at a rate of 1 revolution every 3 seconds has a centrifugal force of 5000 dynes. Find the centrifugal force of a 14 gram object moving in a circle with radius 125 centimeters at a rate of 1 revolution every 2 seconds.